School of Architecture, Science and Technology,
Yashwantrao Chavan Maharashtra Open University

S34121: Physics 01

Programme Code & Name
V92
B.Sc. (PCM)

S34121

Email: director.ast@ycmou.ac.in
Website: www.ycmou.ac.in
Phone: +91-253-2231473

AST, YCMOU, Nashik – 422 222, MH, India

Yashwantrao Chavan Maharashtra Open University

S34121

Physics 01

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Yashwantrao Chavan Maharashtra Open University		
Vice-Chancellor: Prof. Dr. E. Vayunandan		
School of Architecture, Science and Technology		
Director (I/C) of the School: Dr. Sunanda More		
School Council (2018-2020)		
Dr Sunanda More Director(I/c) & Associate Professor, School of Architecture, Science & Technology, YCMOU, Nashik	Dr Manoj Killedar Associate Professor, School of Architecture, Science & Technology, YCMOU, Nashik	Mrs Chetana Kamlaskar Assistant Professor, School of Architecture, Science & Technology, YCMOU, Nashik
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Dr. T.M. Karade Retired Professor, R.T.M. Nagpur University, Nagpur	Mr. D.B. Saundarkar, Representative Study Centre Coordinator, (S.C. Code: 42108) Brahmpuri, Dist. Chandrapur	Dr. D.R. Nandanwar, Joint Director, Technical Education Regional Office, Pune – 411 016

Development Team		
Course Coordinator and Instructional Technology Editor	Book Writer	Book Editor
Dr Manoj Killedar Associate Professor, School of Architecture, Science & Technology, YCMOU, Nashik	Dr Rajendra V Vadnere Director, School of Continuing Education, YCMOU, Nashik	Dr. Sanjay J. Dhoble Prof. Dept. of Physics, R.T.M. Nagpur University, Nagpur

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VICE CHANCELLOR'S MESSAGE

Dear Students, Greetings!!!

I offer cordial welcome to all of you for the Bachelor's degree programme of Yashwantrao Chavan Maharashtra Open University.

As a under graduate student, you must have autonomy to learn, have information and knowledge regarding different dimensions in the field of Physics and at the same time intellectual development is necessary for application of knowledge wisely. The process of learning includes appropriate thinking, understanding important points, describing these points on the basis of experience and observation, explaining them to others by speaking or writing about them. The science of Education today accepts the principle that it is possible to achieve excellence and knowledge in this regard.

The syllabus of this course has been structured in this book in such a way, to give you autonomy to study easily without stirring from home. During the counseling sessions, scheduled at your respective study centre, all your doubts will be clarified about the course and you will get guidance from some qualified and experienced counsellors/ professors. This guidance will not only be based on lectures, but it will also include various techniques such as question-answers, doubt clarification. We expect your active participation in the contact sessions at the study centre. Our emphasis is on 'self-study'. If a student learns how to study, he will become independent in learning throughout life. This course book has been written with the objective of helping in self-study and giving you autonomy to learn at your convenience.

During this academic year, you are required to submit assignments, complete laboratory activities, field visits and the Project work wherever required. You may have to opt for specialization as per programme structure. You will get experience and joy in personally doing above activities. This will enable you to assess your own progress and there by achieve a larger educational objective.

We wish that you will enjoy the courses of Yashwantrao Chavan Maharashtra Open University, emerge successful and very soon become a knowledgeable and honorable Bachelor's degree holder of this university. I congratulate "Development Team" for the development of this excellent high quality "Self- Learning Material (SLM)" for the students. I hope and believe that this SLM will be immensely useful for all students of this program.

Best Wishes!

- Prof. Dr. E. Vayunandan

Vice-Chancellor, YCMOU

FORWARD BY THE DIRECTOR

This book aims at acquainting the students with fundamentals of Physics required at degree level.

The book has been specially designed for Science students. It has a comprehensive coverage of Mathematical concepts and its application in practical life. The book contains numerous mathematical examples to build understanding and skills.

The book is written with self- instructional format. Each chapter is prepared with articulated structure to make the contents not only easy to understand but also interesting to learn.

Each chapter begins with learning objectives which are stated using **Action Verbs as per the Bloom's Taxonomy.** Each Unit is started with introduction to arouse or stimulate curiosity of learner about the content/topic. Thereafter the unit contains explanation of concepts supported by tables, figures, exhibits and solved illustrations wherever necessary for better effectiveness and understanding.

This book is written in simple language, using spoken style and short sentences. Topics of each unit of the book presents from simple to complex in logical sequence. This book is appropriate for low achiever students with lower intellectual capacity and covers the syllabus of the course.

Exercises given in the chapter include MCQs, conceptual questions and practical questions so as to create a ladder in the minds of students to grasp each and every aspect of a particular concept.

The book is presented in two parts. The first part covers the Theory part of the course and aims at improving the basic understanding of the student, with the help of derivations, illustrations from daily life and solved and unsolved examples. Each credit is clearly demarcated. The second part corresponds to the Laboratory portion in the syllabus and is presented in the ready-to-use format of Work Book journal. This part is for two credits and each of the credit blocks is clearly demarcated.

I thank the students who have been a constant motivation for us. I am grateful to the writers, editors and the School faculty associated in this SLM development of the Programme.

Best Wishes !!!

Dr. Sunanda More
Director (I/C) of the School

UNIT 01-01: ORDINARY DIFFERENTIAL EQUATIONS

LEARNING OBJECTIVES

After successful completion of this unit, you will be able to

- Define the differential equations and ordinary differential equations
- Explain the importance of differential equations in physics
- Elaborate on what is meant by homogeneous differential equations
- Discuss the concept of order of homogeneous differential equations
- Solve ordinary differential equations of first and second order with constant coefficients

INTRODUCTION

Have you ever wondered what would have happened if nothing in the world had changed? The universe would have become static. Life would not have existed, and I would not be writing this book nor would you be reading this book!

Since there is a change in the universe, we have physical parameters like velocity which measures how much does a position (vector) of an object changes over a unit time. We have learned in our 12^{th} standard that it is better idea to talk about derivatives rather than unit time to measure the change and hence we define velocity as a derivative of position vector with respect to time, that is, a limit of ratio of infinitesimal change in position (dx) to dt such that dt is infinitely small (tends to zero).

In the system which you may be working the unit of time may be too long and therefore derivative offers a better measure. Let me make it more clear by giving an example. Suppose you are travelling from Nashik to Dhule (around 160 km). The vehicle tells the speed in terms of km per hour. Thus, the unit time is one hour. If you use the definition of distance per unit time, it means that you wait for one hour and find how much distance my car has travelled. Thus, for one hour the speedometer would remain blank, it would be waiting for one hour to be over and then only it would give me my speed. I don't have patience for not having my speedometer to show the reading for one hour. So, what I do is I have mechanism which estimates the distance travelled in a time much smaller than one hour and tell my instantaneous speed (dx/dt). Of course, you may know that vehicles do not actually measure the speed directly but measure a current produced by the vehicle's wheel. The current is calibrated in terms of speed (km/hr).

Thus since the universe is not static, it is continuously evolving. Therefore, we have parameters like speed, acceleration, force, and many more which tell us rate of change of parameters like position, velocity or momentum with respect to time. Many other physical quantities are defined as derivatives with respect to space coordinates (for example you may define force as rate of change of potential with respect to space coordinates).

You may write equations which underline the rate of change of a physical parameter as a function of physical parameters like:

$$\frac{\mathrm{d}Q}{\mathrm{d}x} = f(Q, x)$$

…(01-01)

Here Q is a physical quantity which is dependent on parameter x and f is an arbitaray function. This kind of equation which involves a derivative is a simple example of a differential equation.

In this unit we will be learning about differential equations. In physics, differential equations play very important role. Many of the crucial principles are expressed in the form of differential equations. For example, Coulomb's law of electrostatics, Ampère's law of magnetism, Faraday's law are expressed as partial differential equations which together are called Maxwell's equation. The Maxwell's equations played very important role in understanding electromagnetism. With the help of these equations, the radio waves (which you use in mobile telephony, satellite communication and in internet) were predicted much before they were actually produced in laboratory (by Hertz).

Differential equations are used in other branches of studies also. For example, the problem of finding compound interest on a principle can also be written as a differential equation.

Thus, differential equations are very important in your development as a scientist and as a professional. I am sure you would pay due attention in learning this unit.

01-01: Historical Development of Differential Equation

A differential equation is a mathematical equation that relates some function with its derivatives. In applications, the functions usually represent physical quantities, the derivatives represent their rates of change, and the equation defines a relationship between the two. Because such relations are extremely common, differential equations play a prominent role in many disciplines including engineering, physics, economics, and biology.

In pure mathematics, differential equations are studied from several different perspectives, mostly concerned with their solutions—the set of functions that satisfy the equation. Alternately, the solution to the differential equation is a statement giving the expression for a variable (the dependent variable) which when substituted into the source (given) differential equation leads to an identity. Only the simplest differential equations are solvable by explicit formulas; however, some properties of solutions of a given differential equation may be determined without finding their exact form.

If a self-contained formula for the solution is not available, the solution may be numerically approximated using computers. The theory of dynamical systems puts emphasis on qualitative analysis of systems described by differential equations, while many numerical methods have been developed to determine solutions with a given degree of accuracy.

Let us see the historical development of the concepts in differential equations.

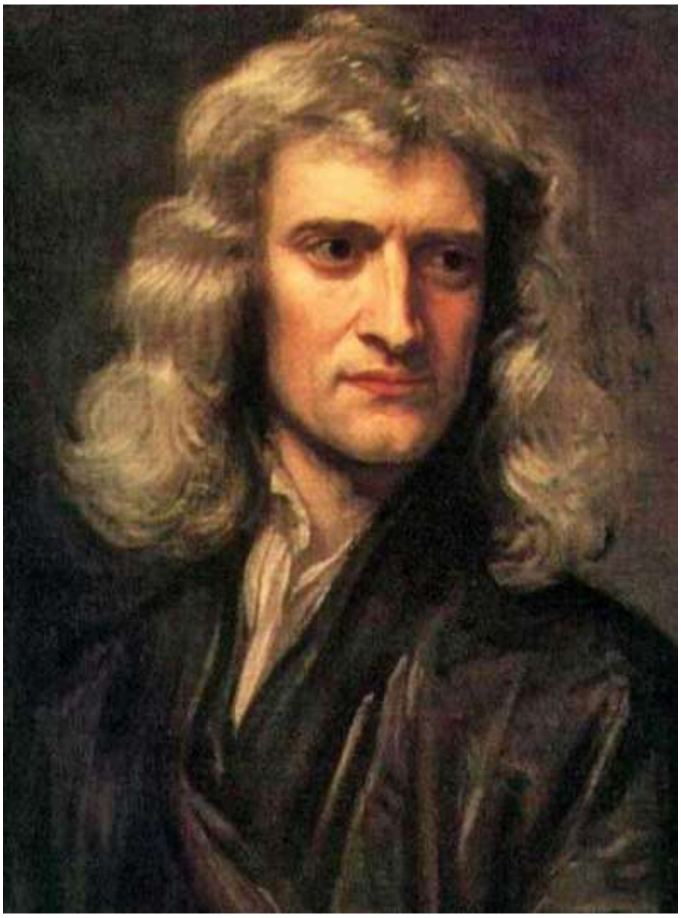


Fig 1.01: Sir Isaac Newton (25 December 1642 – 20 March 1726/27)

(Wikipedia, https://commons.wikimedia.org/wiki/File:GodfreyKneller-IsaacNewton-1689.jpg#/media/File:GodfreyKneller-IsaacNewton-1689.jpg)

Differential equations first came into existence with the invention of calculus by Newton and Leibniz. In Chapter 2 of his 1671 work "Methodus fluxionum et Serierum Infinitarum", Isaac Newton listed three kinds of differential equations:

$$\frac{dy}{dx} = f(x)$$

$$\frac{dy}{dx} = f(x, y)$$

$$x_1 \frac{\partial y}{\partial x_1} + x_2 \frac{\partial y}{\partial x_2} = y$$

… (01-02)

(Here, as you have already studied in 12th standard, $\frac{dy}{dx}$ denotes total differentiation and $\frac{\partial y}{\partial x}$ is the partial derivative of y with respect to x)

He solves these examples and others using infinite series and discusses the non-uniqueness of solutions.

Fig 1.02: Jacob Bernoulli (also known as James or Jacques) (1654 – 1705)

(Wikipedia, https://commons.wikimedia.org/wiki/File:Jakob_Bernoulli.jpg#/media/File:Jakob_Bernoulli.jpg)

Jacob Bernoulli proposed the Bernoulli differential equation in 1695. This is an ordinary differential equation of the form

$$y' + P(x)y = Q(x)y^n$$

...(01-03)

(Here y' is a derivative of y with respect to x)

We will study the Bernoulli equation later in this Unit.

For this, the following year Leibniz obtained solutions by simplifying it.

Fig 1.03: *Portrait de Jean Le Rond d'Alembert* (1717-1783)

(https://commons.wikimedia.org/wiki/File:Alembert.jpg#/media/File:Alembert.jpg)

Historically, the problem of a vibrating string such as that of a musical instrument was studied by Jean le Rond d'Alembert, Leonhard Euler, Daniel Bernoulli, and Joseph-

Louis Lagrange. In 1746, d'Alembert discovered the one-dimensional wave equation, and within ten years Euler discovered the three-dimensional wave equation.

Fig 1.04: Leonhard Euler (15 April 1707 – 18 September 1783)

(https://commons.wikimedia.org/wiki/File:Leonhard_Euler.jpg#/media/File:Leonhard_Euler.jpg)

The Euler–Lagrange equation was developed in the 1750s by Euler and Lagrange in connection with their studies of the tautochrone problem. This is the problem of determining a curve on which a weighted particle will fall to a fixed point in a fixed amount of time, independent of the starting point.

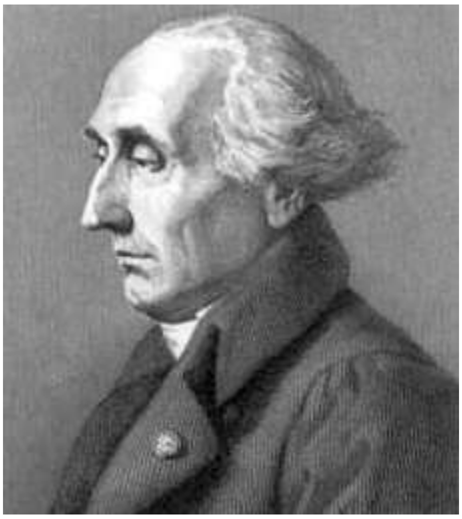


Fig 1.05: Joseph-Louis Lagrange (25 January 1736 – Paris, 10 April 1813)

(https://commons.wikimedia.org/wiki/File:Lagrange_portrait.jpg#/media/File:Lagrange_portrait.jpg)

Lagrange solved this problem in 1755 and sent the solution to Euler. Both further developed Lagrange's method and applied it to mechanics, which led to the formulation of Lagrangian mechanics.

Fig 1.06: Joseph Fourier (21 March 1768 – 16 May 1830)

(https://commons.wikimedia.org/wiki/File:Fourier2.jpg#/media/File:Fourier2.jpg)

Fourier published his work on heat flow in Théorie analytique de la chaleur (The Analytic Theory of Heat), in which he based his reasoning on Newton's law of cooling, namely, that the flow of heat between two adjacent molecules is proportional to the extremely small difference of their temperatures. Contained in this book was Fourier's proposal of his heat equation for conductive diffusion of heat. This partial differential equation is now taught to every student of mathematical physics.

Self-Test 01

(1) Who among the following has been credited of having invented calculus?

(a) Sir Issac Newton

(b) Lagrange

(c) D'Alembert

(d) Euler

(2) If a differential equation cannot be solved to obtain a self-contained formula, which of the following may be done to obtain solution:

(a) It can be solved by using astrology

(b) It cannot be solved and is to be left alone

(c) It can be solved using numerical methods using computers

(e) It can be solved using artificial intelligence methods.

Short Answer Questions 01

(1) What is Newton's contribution in development of Differential Equations?

(2) Elaborate the importance of Differential Equations.

01-02: Types of Differential equations.

Differential equations are the group of equations that contain derivatives. We will study in this Unit, those ordinary differential equations (ODEs) that have simple solutions which contain standard functions like sine, cosine, tangent, e^x, $\ln(x)$, etc. As its name suggests, an ODE contains only ordinary derivatives (no partial derivatives) and describes the relationship between these derivatives of the dependent variable, usually called y, with respect to the independent variable, usually called x. The solution to such an ODE is therefore a function of x and is written $y(x)$. For an ODE to have a "closed-form" solution, it must be possible to express $y(x)$ in terms of the standard elementary functions such as $\exp x$, $\ln(x)$, $\sin x$ etc. The solutions of some differential equations cannot, however, be written in closed form, but only as an infinite series. For some other very complicated DE's, even getting the series solutions may be difficult. We solve such problems using numerical methods on computers.

Ordinary differential equations may be separated conveniently into different categories according to their general characteristics. The primary grouping adopted here is by the **ORDER** of the equation. The order of an ODE is simply the order of the highest derivative it contains. Thus equations containing dy/dx, but no higher derivatives, are called first order, those containing d^2y/dx^2 are called second order and so on. In this Unit we consider first-order and second-order equations. Note that the order does not depend on whether or not you've got ordinary or partial derivatives in the differential equation.

Ordinary differential equations may be classified further according to **DEGREE**. The degree of an ODE is the power to which the highest-order derivative is raised, after the equation has been rationalized to contain only integral powers of derivatives. Hence the ODE

$$\frac{d^3y}{dx^3} + x\left(\frac{dy}{dx}\right)^{3/2} + x^2y = 0,$$

---(01-04)

is of third order and second degree, since after rationalization it contains the term

$(d^3y/dx^3)^2$.

A differential equation is called an **ordinary differential equation**, abbreviated by **ode or ODE,** if it has ordinary derivatives in it. Likewise, a differential equation is called a **partial differential equation**, abbreviated by **pde or PDE,** if it has partial derivatives in it.

The following are examples of ODE:

$$m\frac{dv}{dt} = F(t, v)$$

$$m\frac{d^2u}{dt^2} = F\left(t, u, \frac{du}{dt}\right)$$

… (01-05)

The following are the examples of PDE:

$$\alpha^2 \frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t}$$
$$a^2 u_{xx} = u_{tt}$$
$$\frac{\partial^3 u}{\partial^2 x \partial t} = 1 + \frac{\partial u}{\partial y}$$

...(01-06)

A **linear differential equation** is any differential equation that can be written in the following

form.

$$a_n(t)\, y^{(n)}(t) + a_{n-1}(t)\, y^{(n-1)}(t) + \cdots + a_1(t)\, y'(t) + a_0(t)\, y(t) = g(t)$$

...(01-07)

Here a_n *(t)*, a_{n-1} *(t)* , etc are functions of independent variable *t* and $y^{(n)}(t)$ denotes the n^{th} derivative of *y(t)*.

The important thing to note about linear differential equations is that there are no products of the function, $y(t)$, and its derivatives and neither the function or its derivatives occur to any power other than the first power.

The coefficients like $a_n(t)$ and $g(t)$ can be zero or non-zero functions, constant or non-constant functions, linear or non-linear functions. Only the function, $y(t)$, and its derivatives are used in determining if a differential equation is linear.

Thus, we may also say that a differential equation is defined as linear if it does not contain square or higher power of dependent variable (y) or any of its derivative or product of two.

Let me give an example of linear DE:

$$ay'' + by' + cy = g(t)$$

...(01-08)

Here a,b,c are constants and g is a function of t. Hence it is a Linear DE.

If a differential equation cannot be written in the form, (01-07) then it is called a **non-linear** differential equation. Following is an example of non-linear DE:

$$\sin(y)\frac{d^2y}{dx^2} = (1-y)\frac{dy}{dx} + y^2\mathbf{e}^{-5y}$$

....(01-09)

Why is it a non-linear DE? Because, coefficients of derivatives (namely, sin(y), (1-y)) and coefficient of 1 (i.e., y^2e^{-5y}) are functions of **dependent** variable y and not that of independent variable x.

A differential equation which has no term of constants (that is all the terms are derivatives of variables or the variables themselves) is called a homogenous differential equation. It may be linear or non-linear and may be of any order or of any degree.

For example following are homogeneous DEs :

$$xdx + y\,dy = 0$$

$$\sin(y)\frac{dy}{dx} + \cos(y)\,y = 0$$

Solved Problems

(1)Explain the nature of differential equation

$$\frac{\partial u}{\partial t} + t\frac{\partial u}{\partial x} = 0.$$

...(01-10)

Solution: since it represents a system with two independent variables namely, x and t, and since the derivatives are partial derivatives, the DE is partial.

Similarly as the coefficients are derivatives are 1 and t (dependent variable), hence the DE is linear.

The highest derivative is $\frac{\partial u}{\partial x}$ hence it is first order.

It does not have any constant term hence it is homogeneous.

Thus it is a first order linear homogeneous PDE.

(2) Explain the nature of

$$\frac{d^2u}{dx^2} - x\frac{du}{dx} + u = 0.$$

...(01-11)

Solution: Again, it is not partial (no partial derivatives) hence Ordinary. The highest repeated derivative is second order namely, $\frac{d^2u}{dx^2}$, hence second order. It is linear as maximum "power" or exponent is 1. In other words, there are no terms like $(du/dx)^2$ or $(du/dx)^3$ etc.

It is therefore a second order ODE of second order

SELF-TEST 02

(1) A pde necessarily contains

(a)Partial derivatives

(b)Non-zero coefficient of 1

(c)Second order derivatives

(d)Derivatives of second degree

(2) The degree of a DE is

(a)Coefficient of highest derivative

(b)Highest derivative

(c)Coefficient of the highest exponent of highest derivative

(d)Highest exponent of highest derivative

SHORT ANSWER QUESTIONS 02

(1) How many types of differential equations can be defined?

(2) Elaborate on the concept of non-linear differential equations.

01-03: SOLVING SOME VERY SIMPLE DIFFERENTIAL EQUATIONS

The simplest ordinary differential equations can be integrated directly by finding integrals (anti-derivatives). These simplest odes have the form

$$\frac{d^n x}{dt^n} = G(t),$$

...(01-12)

where the derivative of x = x(t) can be of any order, and the right-hand-side may depend only on the independent variable t. As an example, consider a mass falling under the influence of constant gravity, such as approximately found on the Earth's surface. Newton's law, F = ma, results in the equation

$$m\frac{d^2 x}{dt^2} = -mg,$$

...(01-13)

where x is the height of the object above the ground, m is the mass of the object, and

g = 9.8 meter/sec^2 is the constant gravitational acceleration. The negative sign at RHS is due to the fact that x points "upwards" (i.e., x increases with increasing height) while direction of gravity points "downwards". As the mass cancels from the equation, we have

$$\frac{d^2 x}{dt^2} = -g.$$

...(01-14)

Here, the right-hand-side of the ode is a constant. The first integration, obtained by integration (anti-differentiation), yields

$$\frac{dx}{dt} = A - gt,$$

...(01-15)

Here A is the first constant of integration.

You can interpret this as equation of velocity. If you put t = 0 in this equation, you get A= dx/dt (at t = 0). Thus A is the initial velocity. Hence equation is kinematic equation v = u + at.

We integrate again to get

$$x = B + At - \frac{1}{2}gt^2,$$

...*(01-16)*

Here B is the second constant of integration. The two constants of integration A and B can then be determined from the initial conditions. If we know that the initial height of the mass is x_0, and the initial velocity is v_0, then the initial conditions are

$$x(0) = x_0, \quad \frac{dx}{dt}(0) = v_0.$$

...*(01-17)*

Substitution of these initial conditions into the equations for dx/dt and x allows us to solve for A and B. The unique solution that satisfies both the ode and the initial conditions is given by

$$x(t) = x_0 + v_0 t - \frac{1}{2} g t^2.$$

...*(01-18)*

Solved Problems

(1) Develop the solution to the differential equation for a straight line.

Solution:

A straight line is characterized by a constant slope m:

$$\frac{dy}{dx} = m$$

(01-19)

This is the DE for straight line. Let us verify that it is indeed so.

If we integrate we get,

$$y = mx + C$$

(01-20)

Here C is the constant of integration.

We note the condition that at x = 0, y = C.

Note that when we specify a differential equation along with the value of independent variable y given at the initial value of independent variable x=0, that is y(0), we call such problems as Initial value problems (IVP). In such case constant of integration is found using the information of initial value.

(2) Show that y = A sin(kx) is a solution to the differential equation $\frac{d^2y}{dx^2} + k^2y = 0$

Solution

If we substitute the test solution [y = A sin(kx)] in given DE $\frac{d^2y}{dx^2} + k^2y = 0$,

We get $\frac{d^2y}{dx^2} + k^2y = 0, i.e., -k^2A\sin kx + k^2A\sin kx = 0, i.e., 0 = 0.$

Thus substitution of the test solution leads to an identity. Hence the test solution is indeed the solution to the given DE.

01-04: EXACT DIFFERENTIAL EQUATIONS

A first-order differential equation is one containing a first—but no higher—derivative of the unknown function. For virtually every such equation encountered in practice, the general solution will contain one arbitrary constant, that is, one parameter, so a first-order IVP will contain one initial condition. I will discuss a general method that solves every first-order equation, but there are methods to solve particular types.

I will discuss a type of DE which is called Exact DE. It will be very useful in not only solving many of the DEs but also to understand the general methods of finding solution to the first order DE.

First we will discuss the concept of exact differential.

Given a function $f(x, y)$ of two variables, its exact or **total differential** df is defined by the equation

$$df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy$$

(01-21)

You can understand df to be the change in function f due to changes in both variables x and y.

For example, If $f(x, y) = x^2y + 6x - y^3$, then

$$df = (2xy + 6)\,dx + (x^2 - 3y^2)\,dy$$

The equation $f(x,y) = c$

(01-22)

gives the family of integral curves (that is, the solutions) of the differential equation

$$df = 0$$

(01-23)

Therefore, if a differential equation has the form

$$\frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy = 0$$

(01-24)

For some function $f(x, y)$, then it is automatically of the form $df = 0$, so the general solution is immediately given by $f(x, y) = c$.

In such case,

$$\frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy$$

(01-25)

Is called an **exact differential**, and the differential equation (01-24) is called an **exact equation**.

To determine whether a given differential equation

$$M(x, y)\,dx + N(x, y)\,dy = 0$$

(01-26)

is exact, use the *Test for Exactness*:

A differential equation $M\,dx + N\,dy = 0$ is exact if and only if

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

(01-27)

Remember that the coefficient of dx is partially differentiated with respect to the other variable y and coefficient of dy is partially differentiated with respect to the other variable x.

Why should (01-27) ensure that a function f(x,y) does exist so that (01-24) is valid for it?

If we compare 01-26 with (01-24) we will identify M as $\frac{\partial f}{\partial x}$ and N can be identified with $\frac{\partial f}{\partial y}$.

Thus $\frac{\partial N}{\partial x} = \frac{\partial f}{\partial x \partial y}$ and $\frac{\partial M}{\partial y} = \frac{\partial f}{\partial y \partial x}$.

But since $\frac{\partial f}{\partial x \partial y} = \frac{\partial f}{\partial y \partial x}$, it follows that $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$.

SOLVED PROBLEMS

Example 1: Is the following differential equation exact?

$$(y^2 - 2x)\, dx + (2xy + 1)\, dy = 0$$

Solution

The function that multiplies the differential dx is denoted by $M(x, y)$, so $M(x, y) = y^2 - 2x$; the function that multiplies the differential dy is denoted by $N(x, y)$, so $N(x, y) = 2xy + 1$. Since

$$\frac{\partial M}{\partial y} = 2y \quad \text{and} \quad \frac{\partial N}{\partial x} = 2y$$

the Test for Exactness says that the given differential equation is indeed exact (since $M_y = N_x$). This means that there exists a function $f(x, y)$ such that

$$\frac{\partial f}{\partial x} = M(x, y) = y^2 - 2x \quad \text{and} \quad \frac{\partial f}{\partial y} = N(x, y) = 2xy + 1$$

and once this function f is found, the general solution of the differential equation is simply

$$f(x, y) = c$$

(where c is an arbitrary constant).

Once a differential equation $M\, dx + N\, dy = 0$ is determined to be exact, the only task remaining is to find the function $f(x,y)$ such that $f_x = M$ and $f_y = N$. The method is simple: Integrate M with respect to x, integrate N with respect to y, and then “merge” the two resulting expressions to construct the desired function f.

Example 2: Solve the exact differential equation of Example 1:

$$(y^2 - 2x)\, dx + (2xy + 1)\, dy = 0$$

Solution

First, integrate $M(x,y) = y^2 - 2x$ with respect to x (and ignore the arbitrary “constant” of integration):

$$\int M(x, y)\partial x = \int (y^2 - 2x)\, \partial x = xy^2 - x^2$$

Next, integrate $N(x,y) = 2xy + 1$ with respect to y (and again ignore the arbitrary "constant" of integration):

$$\int N(x,y)\partial y = \int (2xy + 1)\partial y = xy^2 + y$$

Now, to "merge" these two expressions, write down each term exactly once, even if a particular term appears in both results. Here the two expressions contain the terms xy^2, $-x^2$ and y, so

$$f(x,y) = xy^2 - x^2 + y$$

(Note that the common term xy^2 is *not* written twice.) The general solution of the differential equation is $f(x,y) = c$, which in this case becomes

$$xy^2 - x^2 + y = c$$

Example 3: Test the following equation for exactness and solve it if it is exact:

$$x(1 - \sin y)\,dy = (\cos x - \cos y - y)\,dx$$

Solution

First, bring the dx term over to the left-hand side to write the equation in standard form:

$$(y + \cos y - \cos x)\,dx + (x - x\sin y)\,dy = 0$$

Therefore, $M(x,y) = y + \cos y - \cos x$, and $N(x,y) = x - x\sin y$.

Now, since

$$\frac{\partial M}{\partial y} = 1 - \sin y \quad \text{and} \quad \frac{\partial N}{\partial x} = 1 - \sin y$$

the Test for Exactness says that the differential equation is indeed exact (since $M_y = N_x$). To construct the function $f(x,y)$ such that $f_x = M$ and $f_y = N$, first integrate M with respect to x:

$$\int M(x,y)\partial x = \int (y + \cos y - \cos x)\partial x = xy + x\cos y - \sin x$$

Then integrate N with respect to y:

$$\int N(x,y)\partial y = \int (x - x\sin y)\partial y = xy + x\cos y$$

Writing all terms which appear in both these resulting expressions- without repeating any common terms–gives the desired function:

$$f(x, y) = xy + x\cos y - \sin x$$

The general solution of the given differential equation is therefore

$$xy + x\cos y - \sin x = c$$

Example 4: Is the following equation exact?

$$(3xy - y^2)\, dx + x(x - y)\, dy = 0$$

Solution

Since

$$\frac{\partial M}{\partial y} = \frac{\partial}{\partial y}(3xy - y^2) = 3x - 2y$$

but

$$\frac{\partial N}{\partial x} = \frac{\partial}{\partial x}(x^2 - xy) = 2x - y$$

it is clear that $M_y \neq N_x$, so the Test for Exactness says that this equation is not exact. That is, there is no function $f(x,y)$ whose derivative with respect to x is $M(x,y) = 3xy - f^2$ and which at the same time has $N(x,y) = x(x - y)$ as its derivative with respect to y.

Example 5: Solve the IVP (Initial Value Problem)

$$(3x^2y - 1)\, dx + (x^3 + 6y - y^2)\, dy = 0$$
$$y(0) = 3$$

Solution

The differential equation is exact because

$$\frac{\partial M}{\partial y} = \frac{\partial}{\partial y}(3x^2y - 1) = 3x^2 \quad \text{and} \quad \frac{\partial N}{\partial x} = \frac{\partial}{\partial x}(x^3 + 6y - y^2) = 3x^2$$

Integrating M with respect to x gives

$$\int M(x, y)\partial x = \int (3x^2y - 1)\partial x = x^3y - x$$

and integrating N with respect to y yields

$$\int N(x, y)\partial y = \int (x^3 + 6y - y^2)\partial y = x^3y + 3y^2 - \tfrac{1}{3}y^3$$

Therefore, the function $f(x,y)$ whose total differential is the left-hand side of the given differential equation is

$$f(x,y) = x^3y - x + 3y^2 - \tfrac{1}{3}y^3$$

and the general solution is

$$x^3y - x + 3y^2 - \tfrac{1}{3}y^3 = c$$

The particular solution specified by the IVP must have $y = 3$ when $x = 0$; this condition determines the value of the constant c:

$$[x^3y - x + 3y^2 - \tfrac{1}{3}y^3]_{x=0,y=3} = c \Rightarrow 0 - 0 + 27 - 9 = c \Rightarrow 18 = c$$

Thus, the solution of the IVP is

$$x^3y - x + 3y^2 - \tfrac{1}{3}y^3 = 18$$

SOME IN-EXACT DEs CAN BE CONVERTED INTO EXACT DE BY MULTIPLYING BY A FACTOR

Consider the DE

$$x.dy - y.dx = 0$$

(01-28)

Here test of exactness requires that [partial derivative of multiplier of dy wrt x should be equal to partial derivative of multiplier of dx w.r.t. y]. That is:

$$\frac{\partial x}{\partial x} = -\frac{\partial y}{\partial y}$$

(01-29)

This obviously is not satisfied.

If we multiply the given equation (01-28) by x^{-2} we get the following:

$$\frac{dy}{x} - \frac{ydx}{x^2} = 0$$

(01-30)

Here this equation becomes exact, because the new test of exactness requires:

$$\frac{\partial}{\partial x}\left(\frac{1}{x}\right) = -\frac{\partial}{\partial y}\left(\frac{y}{x^2}\right)$$

i.e. $-\frac{1}{x^2} = -\frac{1}{x^2}$

This is an identity.

Thus given equation (01-28) became exact by multiplication by x^{-2}. This factor is called Integrating Factor (IF), which converted the inexact DE into an exact DE.

You can solve this transformed equation using method of exact DE.

You will say, how did you figure out that the factor x^{-2} will work? Well, if you have worked long enough you will find such factors by inspection.

What about other less fortunate guys? Well I will give you a formula for IF. It should work for most cases:

If the differential equation is of the form

$$\frac{dy}{dt} + P(t)y = Q(t),$$

...(01-31)

Then you can verify that it is not an exact DE by rewriting it as

$$dy + \big(P(t)y - Q(t)\big)dt = 0$$

So that condition for exactness becomes [remember, partial derivative of multiplier of dy (here 1) wrt t should be equal to partial derivative of multiplier of dx {i.e., (P-Q)y} w.r.t. t]

$$\frac{\partial 1}{\partial t} = \frac{\partial\big(P(t)y - Q(t)\big)}{\partial y}$$

i.e. 0 =P(t). This is not an identity. Hence the equation is not exact.

How to transform it to an exact DE?

Let

$$I = \int P(t)\,dt$$

(01-32)

If you multiply the given DE (01-31) by e^I it gets converted into an Exact DE.

We will see how it works in the next section. For the time being you may verify that the IF does work.

01-05: HOW DO WE SOLVE A FIRST ORDER LINEAR DIFFERENTIAL EQUATION?

As already explained, we may write a general linear DE as in equation (01-07).

$$a_n(t)y^{(n)}(t) + a_{n-1}(t)y^{(n-1)}(t) + \cdots + a_1(t)y'(t) + a_0(t)y(t) = g(t)$$

If we put n =1 in this we get,

$$a_1(t)y'(t) + a_0(t)y(t) = g(t)$$

...(01-33)

Or, $$\frac{dy}{dt} + P(t)y = Q(t),$$

...(01-34)

Here P(t) and Q(t) are functions of t. These types of equations have been studied in depth and we give you an outline of method to solve equations of this form.

Consider the case of homogeneous first order DE, Q(t) = 0. We get,

$$y' + P(t)y = 0$$

...(01-35)

Or, $$\frac{1}{y}\frac{dy}{dt} = -P(t)$$

...(01-36)

Or, $$\frac{1}{y}dy = -P(t)\,dt$$

...(01-37)

When equation is in this form we say that the variables have been separated. Because on LHS we have terms in dependent variable y alone (no terms of t) and on RHS we have terms which are functions of t only (no terms of y). By integrating we get,

$$\ln y = -\int P(t)\,\mathrm{d}t + C$$

... (01-38)

Here C is constant of integration.

This equation gives

$$y = \exp\left[-\int P(t)\,\mathrm{d}t\right]e^C$$

... (01-39)

Or, $$y = A \cdot \exp\left[-\int P(t)\,\mathrm{d}t\right]$$

...(01-40)

Here A = e^C. Further to simplify the notation we may write,

$$I = \int P(t)\,\mathrm{d}t$$

... (01-41)

For now, let us differentiate I with respect to t:

Then, $\frac{dI}{dt} = P(t)$

...(01-42)

The solution (01-26) of the given differential equation becomes

$$y = A\mathrm{e}^{-\mathrm{I}}$$

... (01-43)

Or $A = ye^{I}$

This function e^{I} is called integrating function (IF). This is so because if we multiply the given equation (01-20) by IF, it becomes exact and we will be able to solve the given first order DE to get a general solution. Let us do it:

Multiplying (01-20) by e^{I}, we get:

$$e^{I}y' + e^{I}P(t)y = Q(t)e^{I}$$

$$e^{I}(y' + P(t)y) = Q(t)e^{I}$$

$$e^{I}\left[\frac{d}{dt}\,(ye^{I})\right] = Q(t)e^{I}$$

i.e

If we differentiate this with respect to t, we get

$$\frac{d(ye^{I})}{dt} = \left(\frac{dy}{dt}\right)e^{I} + y\left(\frac{d\,e^{I}}{dt}\right)$$

$$= y'e^{I} + y\ \ e^{I}I'$$

$$= y'^{e^{I}} + y\,e^{I}P(t)$$

$$\frac{d(ye^{I})}{dt} = e^{I}[y' + P(t)y]$$

(01-44)

If you compare equation (01-20) with the above equation, we will immediately identify the multiplier of e^{I} as Q(t). Thus we can write,

$\frac{d(ye^{I})}{dt} = e^{I}Q(t)$

(01-45)

If we integrate with respect to t, we get:

$$\boldsymbol{y\,e^{I}} = \int \boldsymbol{e^{I}Q(t)dt + C_1}$$

(01-46)

Which gives us,

$$y = e^{-I}\int e^{I}Q(t)dt + e^{I}C_1$$

$$y = [\exp(-\int P(t)dt)\int Q(t)\exp(\int P(t)dt)] + [C_1\exp(-\int P(t)dt]$$

(01-47)

Equation (01-32) is the solution to the differential equation (01-20). The first term $[\exp(-\int P(t)dt)\int Q(t)\exp(\int P(t)dt)]$ is called **Particular integral** and the second term $C_1\exp(-\int P(t)dt)$ is called **Complementary function**.

The Complementary function is the solution to the first order homogeneous differential equation. That is where Q(t) that is the multiplier of 1 (or RHS) is zero. You may check with equation (01-26).

SOLVED PROBLEM

Example 01: Find the solution to the following differential equation.

$$\frac{dv}{dt} = 9.8 - 0.196v$$

Solution

First we need to get the differential equation in the correct form.

$$\frac{dv}{dt} + 0.196v = 9.8$$

From this we can see that *P(t)*=0.196 and so the integrating function (IF) I is then:

$$I = e^{\int P(t)dt} = e^{\int 0.196dt} = e^{t(0.196)}$$

Now multiply all the terms in the differential equation by the integrating factor and do some simplification.

$$\mathbf{e}^{0.196t}\frac{dv}{dt}+0.196\mathbf{e}^{0.196t}v=9.8\mathbf{e}^{0.196t}$$

$$\left(\mathbf{e}^{0.196t}v\right)'=9.8\mathbf{e}^{0.196t}$$

Integrate both sides and don't forget the constants of integration that will arise from both integrals.

$$\int\left(\mathbf{e}^{0.196t}v\right)'dt=\int 9.8\mathbf{e}^{0.196t}\,dt$$

$$\mathbf{e}^{0.196t}v+k=50\mathbf{e}^{0.196t}+c$$

Okay. It’s time to play with constants again. We can subtract k from both sides to get.

$$\mathbf{e}^{0.196t}v=50\mathbf{e}^{0.196t}+c-k$$

Both c and k are unknown constants and so the difference is also an unknown constant. We will therefore write the difference as c. So, we now have

$$\mathbf{e}^{0.196t}v=50\mathbf{e}^{0.196t}+c$$

From this point on we will only put one constant of integration down when we integrate both sides knowing that if we had written down one for each integral, as we should, the two would just end up getting absorbed into each other.

The final step in the solution process is then to divide both sides by $\mathbf{e}^{0.196t}$ or to multiply both sides by $\mathbf{e}^{-0.196t}$. Either will work, but I usually prefer the multiplication route. Doing this gives the general solution to the differential equation.

$$v(t)=50+c\mathbf{e}^{-0.196t}$$

From the solution to this example we can now see why the constant of integration is so important in this process. Without it, in this case, we would get a single, constant solution, $v(t)$=50. With the constant of integration we get infinitely many solutions, one for each value of c.

Example 02

Solve $x^2y' - 2xy = \frac{1}{x}$

Solution

The first step is to write the equation in the form $y' + P(x)y = Q(x)$. For this we divide given equation by x^2:

$$y' + \left(-\frac{2x}{x^2}\right)y = \frac{1}{x^3}$$

Thus $P = -\frac{2}{x}$ and Q=$\frac{1}{x^3}$. The integrating factor I is $\int Pdx = -\int \left(\frac{2}{x}\right) dx = -2\ln(x)$

SELF-TEST 03

(1) For DE, (y cos x)dx + (x cos y) dy =0, test for exactness requires that

(A) cos x = cos y

(B) –y sin x = -y sin y

(C) x = y

(D) y sin x = x sin y

(2) The DE (y cos x)dx + (x cos y) dy = 0 can be transformed to an exact DE by multiplying it by a factor

(A)$\frac{y}{\cos y}\int \frac{\cos x}{x} dx$

(B)exp[$\frac{y}{\cos y}\int \frac{\cos x}{x} dx$]

(C) $\frac{\cos y}{y}\int \frac{x}{\cos x} dx$

(D)exp[$\frac{\cos y}{y}\int \frac{x}{\cos x} dx$]

SHORT ANSWER QUESTIONS 03

(1) Explain the importance of Integrating Factor.

(2) Describe what is meant by an exact DE.

01-04: BERNOULLI EQUATION

A DE of the following form is called a Bernoulli equation:

$$\frac{dy}{dx} + A(x)y = Q(x)y^n \tag{01-48}$$

This equation is a non-linear equation. It can be converted into a linear equation by the following transformation:

$$z = y^{1-n}$$

$$z' = (1-n)y^{n}y'$$

If you multiply both side of the given Bernoulli equation by $(1-n)y^{-n}$, you get

$$(1-n)y^{-n}y' + (1-n)A(x)y^{1-n} = (1-n)B(x)$$

(01-49)

Using the transformation equations, we get,

$$z' + (1-n)A(x)z = B(x)(1-n)$$

(01-50)

This equation is a first order linear DE. You can solve it by using standard techniques.

Solved Problems

Example 1 Solve the following IVP and find the interval of validity for the solution.

$$y' + \frac{4}{x}y = x^3y^2 \quad y(2) = -1, \quad x > 0$$

Solution

So, the first thing that we need to do is- get this into the "proper" form and that means dividing everything by y^2. Doing this gives,

$$y^{-2}y' + \frac{4}{x}y^{-1} = x^3$$

The substitution and derivative that we'll need here is,

$$v = y^{-1} \quad v' = -y^{-2}y'$$

With this substitution the differential equation becomes,

$$-v' + \frac{4}{x}v = x^3$$

So, as noted above, this is a linear differential equation that we know how to solve. We'll do the details on this one and then for the rest of the examples in this section we'll leave the details for you to fill.

Here's the solution to this differential equation.

$$v' - \frac{4}{x}v = -x^3 \qquad \Rightarrow \qquad \mu(x) = \mathbf{e}^{\int -\frac{4}{x}\,dx} = \mathbf{e}^{-4\ \ln|x|} = x^{-4}$$

$$\int \left(x^{-4}v\right)'\,dx = \int -x^{-1}\,dx$$

$$x^{-4}v = -\ln|x| + c \qquad \Rightarrow \qquad v(x) = cx^4 - x^4\ln x$$

Note that we dropped the absolute value bars on the x in the logarithm because of the assumption that $x > 0$.

Now we need to determine the constant of integration. This can be done in one of two ways. We can can convert the solution above into a solution in terms of y and then use the original initial condition or we can convert the initial condition to an initial condition in terms of v and use that. Because we'll need to convert the solution to y's eventually anyway and it won't add that much work in we'll do it that way.

So, to get the solution in terms of y all we need to do is plug the substitution back in. Doing this gives,

$$y^{-1} = x^4\left(c - \ln x\right)$$

At this point we can solve for y and then apply the initial condition or apply the initial condition and then solve for y. We'll generally do this with the later approach so let's apply the initial condition to get,

$$(-1)^{-1} = c2^4 - 2^4\ln 2 \qquad \Rightarrow \qquad c = \ln 2 - \frac{1}{16}$$

Plugging in for c and solving for y gives,

$$y(x) = \frac{1}{x^4\left(\ln 2 - \frac{1}{16} - \ln x\right)} = \frac{-16}{x^4\left(1 + 16\ln x - 16\ln 2\right)} = \frac{-16}{x^4\left(1 + 16\ln\frac{x}{2}\right)}$$

Note that we did a little simplification in the solution. This will help in finding the interval of validity.

Before finding the interval of validity however, we mentioned above that we could convert the original initial condition into an initial condition for v. Let's briefly talk about how to do that. To do that all we need to do is plug $x=2$ into the substitution and then use the original initial condition. Doing this gives,

$$v(2) = y^{-1}(2) = (-1)^{-1} = -1$$

So, in this case we got the same value for v that we had for y. This however doesn't happen in general if problems are solved this way.

Okay, let's now find the interval of validity for the solution. First, we already know that $x>0$ and that means we'll avoid the problems of having logarithms of negative numbers and division by zero at $x=0$. So, all that we need to worry about then is division by zero in the second term and this will happen where,

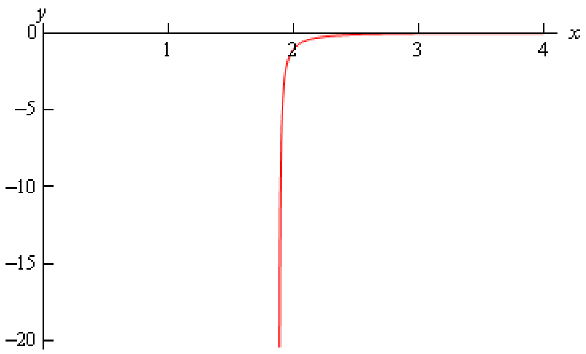
$$\begin{aligned} 1 + 16\ln\frac{x}{2} &= 0 \\ \ln\frac{x}{2} &= -\frac{1}{16} \\ \frac{x}{2} &= e^{-\frac{1}{16}} \qquad \Rightarrow \qquad x = 2e^{-\frac{1}{16}} \approx 1.8788 \end{aligned}$$

The two possible intervals of validity are then,

$$0 < x < 2e^{-\frac{1}{16}} \qquad 2e^{-\frac{1}{16}} < x < \infty$$

and since the second one contains the initial condition we know that the interval of validity is then $2e^{-116}<x<\infty$.

Here is a graph of the solution.



Example 2: Solve the following IVP and find the interval of validity for the solution.

$$y' = 5y + \mathrm{e}^{-2x} y^{-2} \qquad y(0) = 2$$

Solution

The first thing we'll need to do here is multiply through by y^2 and we'll also do a little rearranging to get things into the form we'll need for the linear differential equation. This gives,

$$y^2 y' - 5y^3 = \mathrm{e}^{-2x}$$

The substitution here and its derivative is,

$$v = y^3 \qquad v' = 3y^2 y'$$

Plugging the substitution into the differential equation gives,

$$\frac{1}{3} v' - 5v = \mathrm{e}^{-2x} \qquad \Rightarrow \qquad v' - 15v = 3\mathrm{e}^{-2x} \qquad \mu(x) = \mathrm{e}^{-15x}$$

We rearranged a little and gave the integrating factor for the linear differential equation solution. Upon solving we get,

$$v(x) = c\mathrm{e}^{15x} - \frac{3}{17}\mathrm{e}^{-2x}$$

Now go back to y's.

$$y^3 = c\mathrm{e}^{15x} - \frac{3}{17}\mathrm{e}^{-2x}$$

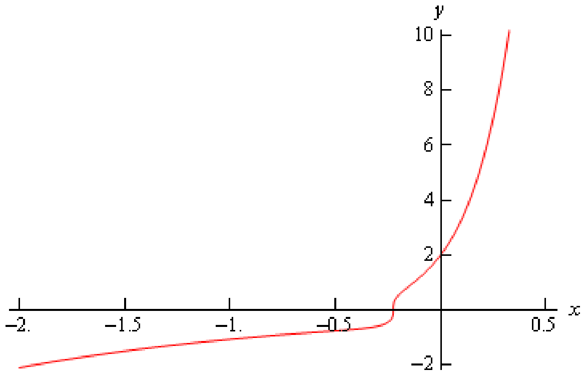
Applying the initial condition and solving for c gives,

$$8 = c - \frac{3}{17} \qquad \Rightarrow \qquad c = \frac{139}{17}$$

Plugging in c and solving for y gives,

$$y(x) = \left(\frac{139\mathrm{e}^{15x} - 3\mathrm{e}^{-2x}}{17} \right)^{\frac{1}{3}}$$

There are no problem values of x for this solution and so the interval of validity is all real numbers. Here's a graph of the solution.



SELF-TEST 04

(1) A Bernoulli Equation is

(A) a non-linear equation which can be converted into linear form

(B) a homogeneous differential equation which can be transformed into inhomogeneous form

(C) an in-exact DE which can be transformed into exact form by multiplying with a Integrating Factor

(D) a partial DE which can be transformed into ordinary DE form

(2) A Bernoulli Equation

(A) can not be solved by analytical methods

(B) can be transformed into linear form and solved using usual methods

(C) can be solved only by using computers

(D) does not correspond to any physical phenomena

01-04: SECOND ORDER LINEAR DIFFERENTIAL EQUATION

In the general DE

$$f_0\,(x)\frac{d^n y}{dx^n} + f_1\,(x)\frac{d^{n-1}y}{dx^{n-1}} + f_2\,(x)\frac{d^{n-2}y}{dx^{n-2}} + \ \ldots + f_{n-1}\,(x)\frac{dy}{dx} + \ f_n\,(x)y = F(x)$$

(01-51)

If we put n = 2, we get

$$f_0\,(x)\frac{d^2 y}{dx^2} + f_1\,(x)\frac{dy}{dx} + \ f_2\,(x)y = F(x)$$

(01-52)

Or

$$y'' + P\,y' + Q\,y = X$$

(01-53)

Here P, Q and X are functions of x.

It is not possible to have a closed solution for DE of order higher than or equal to second order. Many physics problems lead to second order DE with constant coefficient P and Q. We will study the second order DE with constant coefficient P and Q.

01-04: Solving the Second order linear differential equation with constant coefficients

Homogeneous Linear Equation

When we have F(x) =0, and P, Q as constants, we will get homogeneous linear DE.

$$\frac{d^2y}{dx^2} + A_1\frac{dy}{dx} + A_2\,y = 0$$

(01-54)

Here A_1 and A_2 are constants.

We denote the operation of differentiation by D≡ $\frac{d}{dx}$

Thus, $\frac{dy}{dx}$ can be denoted by Dy.

Similarly, $\frac{d^2}{dx^2}$ being same as $\frac{d}{dx}\left(\frac{d}{dx}(y)\right)$ can be denoted by D^2y.

Hence equation (01-54) can be written as

$$(D^2 + A_1D + A_2)y = 0$$

For the sake of convenience it is also written as:

$$(D^2 + A_1D + A_2) = 0$$

Or

$$F(D) = 0$$

(01-55)

You may have already noted that this equation is a quadratic equation in D and hence two roots in general.

The root of F(D)= 0 can be real, distinct, repeated or complex. Let us discuss these cases separately

REAL AND DISTINCT ROOT

When the roots are real and distinct, you can denote them as m_1 and m_2. The general solution of the given second order linear DE is

$$y = C_1 e^{m_1 x} + C_2 e^{m_2 x}$$

(01-56)

REAL AND REPEATED (EQUAL) ROOTS

If roots are m_1=m_2=m, equation (01-54) would give:

$$y = C_1 e^{mx}$$

Which suffers from the defect that only one constant (of integration) is present which is not acceptable for a second order DE.

We can write (D-m)(D-m)y =0. If we have (D-m)y =v, then

(D-m)v = 0

Or $\frac{dv}{dx} - mv = 0$

The solution to this is $v = Ae^{mx}$.

Hence, $(D - m)y = Ae^{mx}$, or

$\frac{dy}{dx} - my = Ae^{mx}$

This is first order linear DE. We know how to solve such equations. Using the method described in earlier section, we get the solution as

$ye^{-mx} = \int e^{-mx} Ae^{mx} dx = \int Adx = Ax + B$

The total solution will be:

$$y = (Ax + B)e^{mx}$$

(01-57)

COMPLEX CONJUGATE ROOTS

We know from the theory of quadratic equation that the roots of a quadratic equation, when they are not real, come in pair of complex numbers. Such pair is called complex conjugate and is of the form $\alpha \pm \beta i$. Thus the DE can be written as

$$(D - \alpha - i\beta)(D - \alpha + i\beta)y = 0$$

Since the roots are not equal, the solution (as described earlier in distinct roots case)

$$y = C_1 e^{m_1 x} + C_2 e^{m_2 x}$$

$$m_1 = \alpha + i\beta \text{ and } m_2 = \alpha - i\beta$$

$$y = C_1 e^{(\alpha + i\beta)x} + C_2 e^{(\alpha - i\beta)x}$$

$$y = e^{\alpha x}\left[C_1 e^{(i\beta)x} + C_2 e^{(-i\beta)x}\right]$$

You may know that,

$$e^{(i\beta)} = Cos\beta + \mathrm{i}\,\mathrm{Sin}(\beta) \text{ and } e^{(-i\beta)} = Cos\beta - \mathrm{i}\,\mathrm{Sin}(\beta)$$

$$y = e^{\alpha x}[(C_1 + C_2)Cos(\beta x) \; + i(C_1 - C_2)\sin(\beta x)]$$

Since C_1 *and* C_2 are constants their additions and subtractions will be constants too.We can write the solution as:

$$y = e^{\alpha x}[ACos(\beta x) \; + iB\sin(\beta x)]$$

(01-58)

Here A and B are the constants of integration.

Another form of the solution is

$$y = Ce^{\alpha x}\sin(\beta x + D)$$

(01-59)

Here C and D are the arbitrary constants.

Solved Problems

Example 1 Determine some solutions to y′′−9y=0

Solution

We can get some solutions here simply by inspection. We need functions whose second derivative is 9 times the original function. One of the first functions that comes back to itself after two derivatives, I think, is an exponential function and with proper exponents the 9 will get taken care of as well.

So, it looks like the following two functions are solutions.

$$y(t) = \mathbf{e}^{3t} \qquad \text{and} \qquad y(t) = \mathbf{e}^{-3t}$$

We'll leave it to you to verify that these are solutions in fact

However, these two functions are not the only solutions to the differential equation. Anyof the following are also solutions to the differential equation.

$$y(t) = -9\mathbf{e}^{3t} \qquad y(t) = 123\mathbf{e}^{3t}$$
$$y(t) = 56\mathbf{e}^{-3t} \qquad y(t) = \frac{14}{9}\mathbf{e}^{-3t}$$
$$y(t) = 7\mathbf{e}^{3t} - 6\mathbf{e}^{-3t} \qquad y(t) = -92\mathbf{e}^{3t} - 16\mathbf{e}^{-3t}$$

In fact if you think about it, any function that is in the form

$$y(t) = c_1\mathbf{e}^{3t} + c_2\mathbf{e}^{-3t}$$

will be a solution to the differential equation.

Example 2: Solve the following IVP.

$$y'' + 11y' + 24y = 0 \qquad y(0) = 0 \qquad y'(0) = -7$$

Solution

The characteristic equation is (we are writing r for D)

$$\begin{aligned} r^2 + 11r + 24 &= 0 \\ (r+8)(r+3) &= 0 \end{aligned}$$

Its roots are $r_1=-8$ and $r_2=-3$ and so the general solution and its derivative is.

$$\begin{aligned} y(t) &= c_1\mathbf{e}^{-8t} + c_2\mathbf{e}^{-3t} \\ y'(t) &= -8c_1\mathbf{e}^{-8t} - 3c_2\mathbf{e}^{-3t} \end{aligned}$$

Now, plug in the initial conditions to get the following system of equations.

$$\begin{aligned} 0 &= y(0) = c_1 + c_2 \\ -7 &= y'(0) = -8c_1 - 3c_2 \end{aligned}$$

Solving this system gives $c_1=7/5$and $c_2=-7/5$. The actual solution to the differential equation is then

$$y(t) = \frac{7}{5}\mathbf{e}^{-8t} - \frac{7}{5}\mathbf{e}^{-3t}$$

Short Answer Questions 04

(1) Explain the importance of auxiliary equations

(2) Explain how a Second order DE with constant coefficients, which has an auxiliary equation (or characteristic equation) giving complex conjugate roots can be solved.

Key Words

Differential equation, partial differential equation, exact DE, integrating factor, Bernoulli Equation, Auxiliary equation (or characteristic equation)

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Course Companion Website

Visit Here for Course Companion website for this course:

UNIT 01-02: LAWS OF MOTION

LEARNING OBJECTIVES

After successful completion of this unit, you will be able to

- Describe frames of reference
- Elaborate concepts of Newton's Laws of motion
- Describe the dynamics of a system of particles
- Explain the concept of centre of mass

INTRODUCTION

Fig 2.01: Vienna regulator style pendulum wall clock

(Source:**https://commons.wikimedia.org/wiki/File:GB-3-Gew-Pendeluhr_(Luekk).jpg#/media/File:GB-3-Gew-Pendeluhr_(Luekk).jpg)**

You may have seen a pendulum which used to be a common thing during your grandfather's time. It used to be on walls of most wealthy persons in the town. You can see one such clock in the adjacent figure. You can hardly miss the heavy bob of the pendulum. It plays very important part in time keeping. The time period of the oscillation depends only on the length of the pendulum (distance between point from which it is hanging to the centre of bob) at a given place. Therefore it could accurately control the mechanism of the clock and help keep the time.

Suppose you want to study the motion of a pendulum suspended from a nail on the wall using a string. In this case you are using a coordinate system such that the nail serves as the 'origin' and the angle which is made by the string is the parameter which tells you the position of the bob at any given time. You may use a stop watch to measure the time. Whenever you start a stopwatch, you are marking the time as 'zero'. We may say that the origin of the time is being adjusted. Thus, if you collect the information on angle versus the time on stop watch, you have mapped the motion of the pendulum.

The coordinate system which is physically represented by the nail as the point of origin, the string which helped you measuring the angle and the time measured using the clock is called a frame of reference. We will study the frame of reference in greater details in this unit.

The frame of reference is the most fundamental tool in measuring motion of a body. Once we have learned about it, we may move on to the motion itself. The laws of motions were formulated by Newton which you may have learned in high school. We will revisit them.

Mathematics is an abstract subject, which is very useful in understanding the physical phenomena in nature. We have to map the physical phenomena to the world of mathematics. For example 'point' is an abstract mathematical concept. You can never see a 'point'. Why?

A point is an exact position or location on a plane surface. It is important to understand that a point is not a thing, but a place. We indicate the position of a point by placing a dot with a pencil. This dot may have a diameter of, say, 0.2mm, but a point has no size. A dot made using pencil is a 'approximation' of the point. A point has zero size and hence can't be seen. To make it visible you use a dot. But we should remember that a dot is not a point.

A physical object like the bob of the pendulum is a large object. But in order to make use of the concepts of mathematics we have to assign a 'point' to describe its motion.

Which part of the bob is ideal for this purpose? Is it the top part or the bottom part or left end?

This question leads us to the concept of centre of mass for a physical system of many particles (like pendulum bob) which moves together. We will study these concepts as well.

02-01: Frame of Reference

If we want to describe the motion of an object, we always have to assume existence of a co-ordinate system, relative to which the motion will be described, because motion is always relative. For example if a train is travelling at speed of 60 km/h, it is necessary to say that it is so relative to the station near which it is passing. On the other hand, if another train is moving on a parallel track in the same direction with equal speed, the speed of the train relative to other train will be zero. Thus, with respect to the station the speed of the train is 60 km/hr, while with respect to the second train it is zero. It means whenever you talk about the velocity of an object, you have to specify with respect to which object you have measured the velocity.

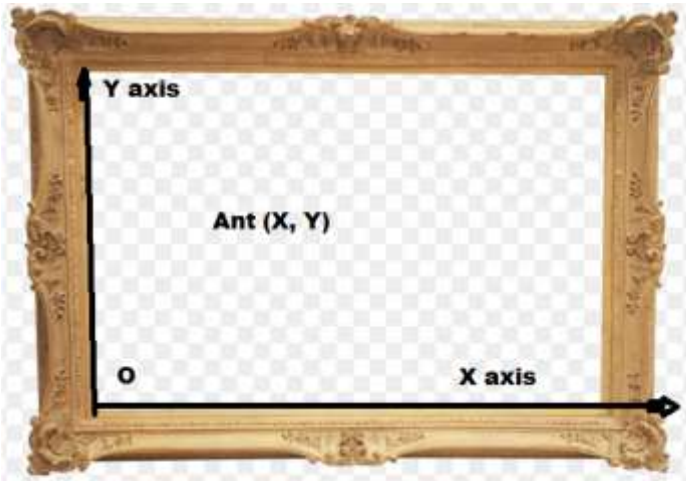


Fig 2.02: A photo-frame as a reference frame. You can label lower left corner as Origin, lower edge as X axis, Left vertical edge as Y axis to describe the motion of an ant on it.

(Source:https://www.kisspng.com/png-victorian-era-picture-frames-bed-frame-framing-ant-820143/)

We need to give a name to this phenomenon (that “the motion is relative”). We call it ‘Relativity’. We also need to give name to the system with respect to which the motion is measured: we call it ‘frame of reference’. To digest this concept easily, I used to visualize a frame (like a photo-frame) which is fitted on an object like the seat of my car. I further visualize an ant moving on the photo-frame. This would describe the motion of the ant on the photo-frame. If the lower left corner of the photo-frame is labeled as origin (O) and horizontal lower plate of the frame is labeled as X-axis and vertical plate on left is called Y-axis, this photo frame serves my co-ordinate system. The motion of the ant is described in the frame of reference fixed in the car. Hence velocity of car in this frame is zero (car is stationery in this frame). But the ant may be moving in this frame. I hope I have made my point clear.

Let me get to the formal definition now: A system relative to which the motion of any object is described is called a frame of reference.

In physics, a frame of reference (or reference frame) consists of an abstract coordinate system and the set of physical reference points that uniquely fix (locate and orient) the coordinate system and standardize measurements.

In n dimensions, n+1 reference points are sufficient to fully define a reference frame. Using rectangular (Cartesian) coordinates, a reference frame may be defined with a reference point at the origin and a reference point at one unit distance along each of the n coordinate axes.

In Einsteinian relativity, reference frames are used to specify the relationship between a moving observer and the phenomenon or phenomena under observation. In this context, the phrase often becomes "observational frame of reference" (or "observational reference frame"), which implies that the observer is at rest in the frame, although not necessarily located at its origin. A relativistic reference frame includes (or implies) the coordinate time, which does not correspond across different frames moving relatively to each other. The situation thus differs from Galilean relativity, where all possible coordinate times are essentially equivalent.

KINEMATICS OF FRAMES OF REFERENCE

Frame of Reference is one of the basic concepts in Kinematics. Kinematics being based upon frame of reference provides a great deal about the nature of motion of a body in a given sequence of circumstances. Physics involves the study of our surroundings, a form of analysis where everything is put through a logical reasoning process and often provided a theory for. Motion of object around us like cars, rockets, and plane can be understood through this concept. Relative motion or motion of objects with respect to each other is also described using this concept.

To analyze the motion of a body, first we need to find the location of the body at a certain instant of time. To do this, we have the coordinate system. The system of X, Y and Z axes, which are perpendicular to each other, are used to mark the position of an

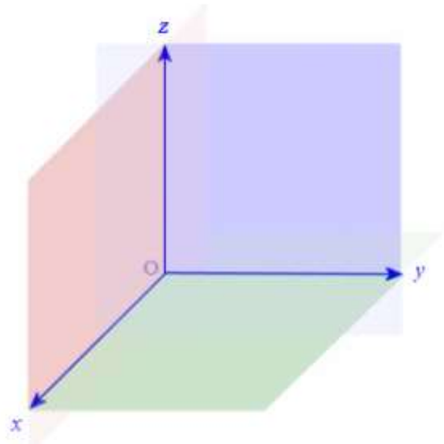


Fig 2.03: Cartesian coordinate system in three dimensions

(Source: https://www.toppr.com/bytes/frame-of-reference-physics/)

object in a 3 dimensional space. This frame of 3 axes is the frame of reference, which originates from the interaction of the three axes which is the origin. Now in this 3 dimensional space, we can analyze the position and thus the motion of the object. We can predict its path if it follows a path pattern described periodically.

WHAT IS FRAMES OF REFERENCE IN PHYSICS?

Imagine you threw and caught a ball while you were on a train moving at a constant velocity past a station. To you, the ball will simply go vertically up and then down under the influence of gravity. However, if there's an observer on the station platform, then that person will see the ball travel in a parabola, with a constant horizontal component of velocity equal to the velocity of the train.

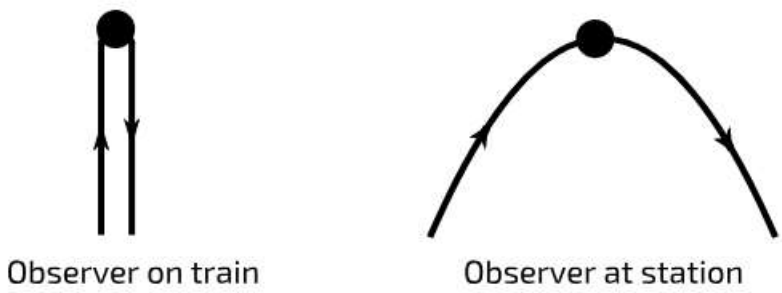


Fig 2.04: Path of a body (ball) for observer in the train is different than that of the observer at station.

(Source: https://isaacphysics.org/concepts/cp_frame_reference)

So, the different observations occur because the two observers are in different frames of reference. A frame of reference is nothing but a set of coordinates that can be used to determine positions and velocities of objects in that frame; different frames of reference move relative to one another.

Simply put, when you are standing on the ground, that is your frame of reference. Anything that you see, watch, or measure will be compared to the reference point of the ground. If you are standing at the back of a moving truck, the truck is now your frame of reference, and everything will be measured, compared to it.

There are two types of frames of reference: Inertial and non-inertial frames.

Inertial Frame of Reference

A frame of reference that remains at rest or moves with constant velocity with respect to other frames of reference is called Inertial Frame of Reference. An inertial frame of reference has a constant velocity. That is, it is moving at a constant speed in a straight line, or it is standing still. Newton's laws of motion are valid in all inertial frames of reference. Here, a body does not change due to external forces. All inertial frames of reference are equivalent for the measurement of physical phenomena.

There are several ways to imagine this motion:

- Our earth.
- A space shuttle moving with constant velocity relative to the earth.
- A rocket moving with constant velocity relative to the earth.

Non-inertial Frame of Reference

A frame of reference is said to be a non-inertial frame of reference when a body, not acted upon by an external force, is accelerated. In a non-inertial frame of reference, Newton's laws of motion are not valid. It also does not have a constant velocity and is accelerating. There are several ways to imagine this motion:

- The frame could be travelling in a straight line, but be speeding up or slowing down.
- The frame could be travelling along a curved path at a steady speed.
- The frame could be travelling along a curved path and also speeding up or slowing down.

To get a detailed idea about the concepts and solve mind-boggling problems in Kinematics, you can try reading leading author HC Verma's Concept of Physics in class 11th and 12th.

Solved Problems

Example (1) One dimensional uniform velocity problem

Consider two particles, A and B. Let the laboratory be where the measurements are made and it is stationary. In the laboratory frame denoted by S, A moves left-to-right with speed $u_A = 5ms^{-1}$ and B moves left-to-right with speed $u_B = 3ms^{-1}$. This is illustrated in Figure below.

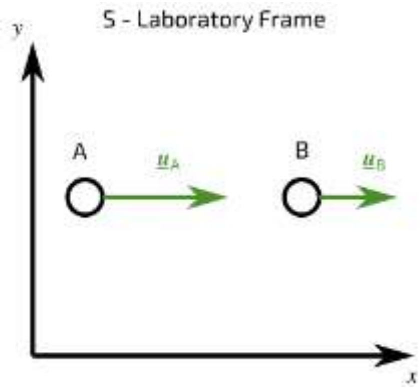


Fig 2.05: Lab frame of reference for problem

In the rest frame of A, A has speed $0ms^{-1}$, so this frame, which we could denote S', must move at the same speed as A. The speed of S' relative to the laboratory frame is thus u_A. To make things easier, we choose the axes in the two frames to point in the same directions, and for them to coincide at time t=0s.

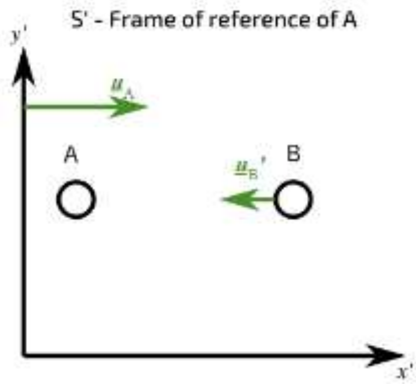


Fig 2.06: Motion of A and B in the reference frame of A (i.e., in which A is at rest)

We can transform from one frame to another by adding or subtracting their relative velocity, depending on the relative direction of the relative motion. Thus, to transform from the laboratory frame to S' in the example above, one can subtract u_A from the velocities of the particles: the speed of A becomes $u'_A = u_A - u_A = 0 ms^{-1}$ (this is how we defined S'), and that of B becomes $u'_B = u_B - u_A = 3-5 = -2 ms^{-1}$. This means that in the rest frame of A, B moves right-to-left with speed 2ms−1. The particles in S' are illustrated in Figure 2.06

You can find the velocities of A and B in the reference frame of B yourselves.

EXAMPLE (2) TWO DIMENSIONS UNIFORM VELOCITY PROBLEM

Of course, objects are not always moving in one dimension along the same line. Here it's easiest to think of subtracting a velocity **v** as being equivalent to adding a velocity –**v**. This is demonstrated with another two objects, C and D, moving in different directions in a reference frame S as shown in Figure.

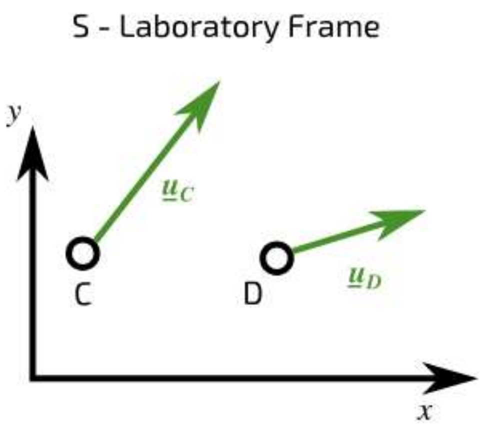


Fig 2.07: Particles motion in 2D lab frame of reference.

To move into the frame of reference of C, S', we must subtract u_C from all of the velocities in frame S. This gives $u'_C = u_C - u_C = 0$ as expected, and $u'_D = u_D - u_C$. This is shown below in Figure

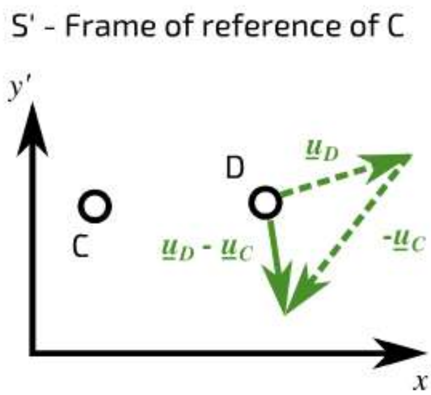


Fig 2.08: Velocities in Frame of C.

I leave it to you to do the transformation for Frame of D.

In lab frame of reference the velocities of particle A and B is (2,3,4) and (10,12,13). What will be the velocity of B in which A is stationary?

Solution:

Here u_A = (2,3,4) and u_B = (10,12,13) in Lab frame S.

Hence in frame S', velocities can be found by adding $-u_A$ from the velocities in S: $u'_A = u_A - u_A = 0$, $u'_B = u_B - u_A = (10,12,13) - (2,3,4) = (8,9,9)$.

I leave it to you to find the Velocities of A and B for Frame of B.

Example (3) The elevator problem (one dimension accelerating systems)

Consider a case similar to the ones discussed in previous two examples. This time, consider accelerating systems.

(a) An open lift is moving with an acceleration a = 1.20 m/s^2 in upwards direction with respect to the person sitting on ground in Lab frame. The man in lift drops a ball out of the lift by releasing it when the lift was 4.00 m from ground. What will be the speed of a ball (just before it touches ground) in Lift's frame and in Lab's frame?

(b) The said lift is falling freely under gravity, i.e., it is moving with an acceleration a = g = 9.80 m/s^2 in downwards direction with respect to the person sitting on ground. The man in lift drops a ball out of the lift by releasing it when the lift was 4.00 m from ground. What will be the speed of a ball dropped by the man in the lift in the two frames?

Strategy and Concepts

In Lab frame (stationary person on ground), the speed of lift is given by

$$v = u + a\,t$$

We are assuming upward direction as positive and downward direction as negative.

The acceleration of the ball (in Lift frame) will be the vector addition of the acceleration a and gravitational acceleration –g. Effective or net acceleration = a – g = 1.20-9.80 m/s^2 =-8.60 m/s^2. Negative sign implies that acceleration is pointing downwards. The man in lift will see the ball as if it has a initial velocity of zero at the time of release and then it is subjected to an acceleration=-8.60 m/s^2.

For part (b) we can simply put a=-g.

Solution

Part (a)

Let us identify the known and unknown parameters.

Let x_B or $x_{B(t)}$ be the position of the ball on Lift as seen in Laboratory frame.

Let t =0 when Lift is on ground and is making upwards journey. We set our coordinates so that x_B is at zero mark.

At t =0, x_B =0.

At t = t_1, the Lift reaches distance 4.00m (and man drops ball). So

At t = t_1, x_B =4.00m

At t = t_2, the ball reaches ground, x_B =0.

We have to find velocity of ball $v_{B(t2)}$ at t =t_2.

In the Lab frame, the ball will be seen as if it is imparted an initial velocity of v (the velocity of lift at the time of release of ball) and then the ball is subject to the gravitational acceleration

If the person is at distance 4.00 m from ground when he dropped the ball, the speed of ball in Lab frame at the time of release can be found first by finding time t1 which took the lift to move the distance 4.00m from ground. Then we put the value of time in first kinematic equation (v = u + at) to get the velocity

$$4.00 = ut + \frac{1}{2}\, a\, t^2 = \frac{1}{2}\,(1.20)t^2$$

From this relation we find t_1 as 2.58 s. We get $v_{B(t1)} = 0 + 1.20*2.58 = 3.098$ m/s.

The ball is subjected to initial velocity of 3.098 m/s (upwards) and them it is subjected to an acceleration of 9.80 m/s^2 (downwards) as seen in the Lab frame. In the Lab frame the ball will be seen as if it is moving upwards (with initial speed 25.30 m/s) under gravity, attains a maximum height and then starts downwards journey.

It can be demonstrated that when a ball is thrown with a speed V_1 upwards, it returns back with same speed V_1 with direction flipped. So the ball which went up at time t_1 with speed 3.098 m/s, will return after some time at the same spot (in Lab frame) with speed 25.30 m/s.

If I make use of this fact, I can find the final velocity using third kinematic equation ($v^2 = u^2 + 2as$)

Put u = 3.098, a = 9.80 m/s/s, and s = 4.00, remember that direction of u and a is same (downwards) :

$$v = \sqrt{(3.098)^2 + 2 \times 9.80 \times 4.00}$$

$$v = 9.38\, ms^{-1}$$

In Lift frame,

$$s = ut + \frac{1}{2}(a - g)t^2$$

Here s is the distance from ground (4.00 m), initial velocity u =0, t is the time to reach ground (to be found)

$$t = \sqrt{\left(2 \times \frac{s}{a-g}\right)} = \sqrt{2 \times \frac{4}{8.60}} = \sqrt{0.9302} = 0.964 \text{ s}$$

In time 0.964 second, the speed attained by the ball (just before it hits the ground under acceleration of 8.60 m/s^2, will be

$$v = 0 + 8.60 \times 0.964 = 7.715 \text{ m/s}^2$$

I have plotted the position of ball and that of Lift with respect to time in adjacent figure.

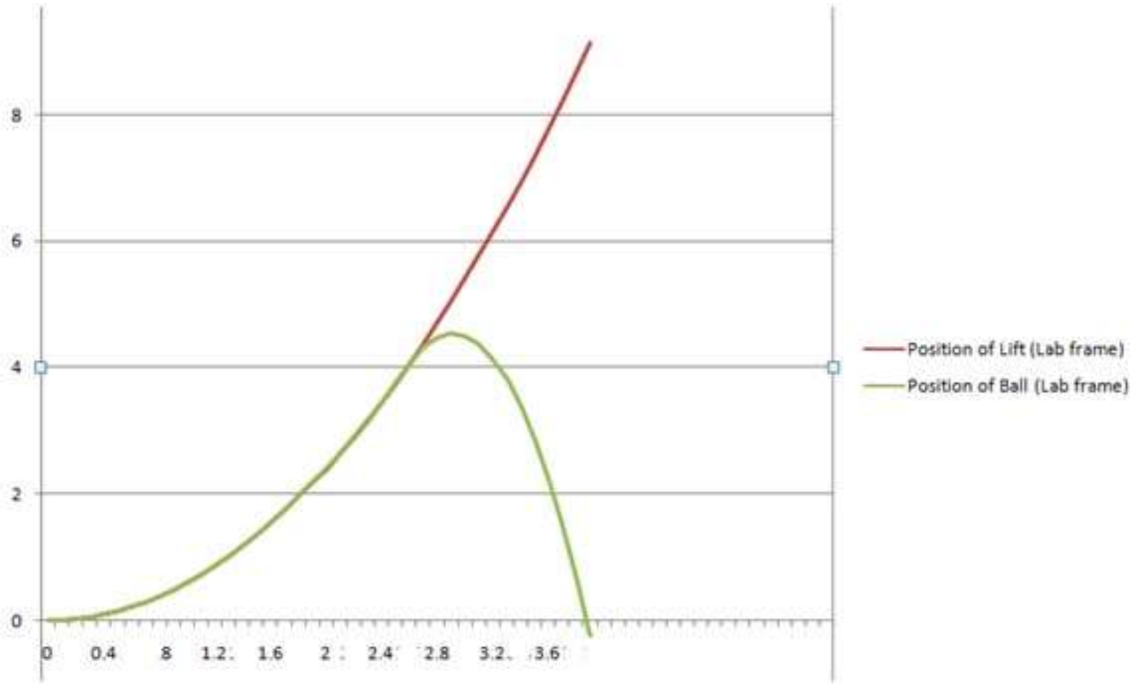


Fig 02-09: Plot of position of Lift and that of ball with respect to time. Note that lift and ball coincide (with acceleration of 1.2 ms^{-2}) till the Lift and ball reach 4.0 m height, then as the ball is dropped, the ball moves upwards due to speed gained by movement of lift, then it starts moving under influence of gravity. X-axis is time in s, Y axis is position in m.

Part (B)

Let me collect the known and unknown variables:

At t = 0, the ball is at distance $x_B = 4.00$

At $t = t_1$, the ball hits the ground, $x_{B(t1)} = 0$.

At t =0, $v_B = 0$ (Lab frame)

At $t = t_1$, v_B =?

In Lab frame, the ball is travelling with acceleration g (downwards) and so is the frame itself. Hence there is no relative motion for the ball in the Lift frame of reference.

In Lab frame, the final velocity is given by the third kinematic equation

$$v^2 = u^2 + 2as$$

Here u =0, a= 9.80 ms-2, s = 4.00m.

$v = \sqrt{2 \times 4.00 \times 9.80}$ = 8.85 m/s.

In the Lift frame of reference the velocity of ball =0.

Discussion

Here we have seen very strange result. When the lift is undergoing the free-fall, there was no relative motion of the ball in the Lift frame. It appeared as if there was no gravity. This is a very important result. We will get back to the elevator problem again.

SELF-TEST 05

(1)Which of the following is example of inertial frames of reference?

(A) Motion of the earth relative to the moon

(B) Motion of a truck with a constant speed in straight line relative to a standing person

(C) Motion of a cyclist moving in circular orbit with uniform angular velocity

(D) Motion of the spinning galaxy seen from earth

(2) Which of the following is example of non-inertial frames of reference?

(A) A train at rest with respect to a person standing on platform

(B) A train at uniform speed in a straight line with respect to a person standing on platform

(C) A planet moving in elliptical orbit around the sun

(D) Two trains moving with uniform speeds of 55 kmph and 120 kmph along straight paths parallel to each other.

Short Answer Questions 01

NASA has two ground-based facilities for studying the effects of apparent weightlessness — the above ground 2.2 Second Drop Tower and the subterranean 5 Second Drop Tower (a.k.a. the Zero-Gravity Research Facility). Both are located near Cleveland, Ohio and are associated with NASA's Glenn Research Center. Both are long vertical rooms where experiment packages are dropped in a minimal air resistance environment, briefly simulating weightlessness. The table below gives two key characteristics for each drop tower. Determine the other remaining characteristics and complete the table.

NASA drop towers

Characteristic	2.2 s drop tower	5 s drop tower
overall length (m)		
free fall duration (s)		5.18
free fall distance (m)	24.1	
impact velocity (m/s)		
deceleration duration (s)	0.2	
deceleration distance (m)		
impact deceleration (m/s2)		
impact frame of reference (g)		35

02-02: Newton's Laws of Motion

We usually begin the study the motion of a particle using simple kinematics equations in like

$\mathbf{v} = \mathbf{u} + \mathbf{a}t$

$\mathbf{s} = \mathbf{u}t + 1/2\ \mathbf{a}\ t^2$

$\mathbf{v}^2 = \mathbf{u}^2 + 2\ \mathbf{a.s}$

Here t is time, **u** is the initial velocity (at t = 0), **v** is final velocity at t = t, **a** is acceleration and s is the distance covered. We have used bold character to denote vector quantities and dot is used for scalar product.

When we do this type of analysis we did not ask what caused the motion. Our analysis is largely geometrical. The aspect of mechanics which deals with the study of what caused the motion is called *dynamics*.

For our convenience we may consider a point to represent a body under study. For example if you are studying the motion of a canon or bullet fired at angle α with initial velocity v, the canon or bullet is the 'particle'. The earth which asserts a gravitational pull in the form of g can be identified as 'environment'. The *particle* denotes the body whose motion we are studying and *environment* denotes the entity which influences the motion.

When we consider the motion of a planet which is orbiting the sun, we consider planet as 'point' and the sun as the environment.

Similarly if you are observing the motion of a bar magnet A under the influence of the magnetic field generated by a solenoid S, the magent A is *particle* and solenoid S is *environment*.

In this unit we will be studying those cases in which the speed of the particle is small compared with the speed of light in vacuum ($c = 3 \times 10^{10} m\, s^{-1}$). This is called the domain of *classical mechanics*. We will study the cases when speed of particle is comparable to c in Unit 4 under *Special theory of Relativity*.

The central problem of classical mechanics can be summarized as follows:

(1) You are given a particle whose characteristic parameters like mass, charge, dipole moment, etc are known to you

(2) You place such particle under influence of the environment (whose complete description is known to you) and you know particle's initial velocity (i.e. velocity at a time which we mark as zero). This forms the 'system'.

Fig 2.10: Newton's laws of motions determine the path of dolphin's motion

(credit: Jin Jang))

(3) You have to find the motion of the particle at subsequent times t (i.e., find position of particle at all the times subsequent to t = 0).

This problem was solved by Issac Newton for a substantial range of system through his Laws of Motion and Law of Universal Gravitation. The outline of the plan to solve the central problem in classical mechanics is as follows:

(1) We define concept of force **F**, which gives us the acceleration **a** experienced by a "standard" particle. For example you may consider a block of 1 cm cube of water to be a standard ('Unit') particle. You may use such standard particle for purpose of calibration.

(2) We develop a procedure for assigning mass *m* to a body so that we understand that different bodies may have different acceleration (due to their differing masses) under the same environment. For example the mass m assigned to block of 1 cm cube of water may be identified as 1 g. Thus you can observe that two blocks of 1 cm cube have acceleration (under say free fall) which is half that for one block.

(3) We may find forces that act on particles using properties of environment and that of the particle, using what will call as *force laws*. For example if the environment consists

of an inclined plane you derive the formula for modified force along the direction of motion ($mg\ cos\alpha$). We use this information to calculate the equation of motion.

Thus Force is at the core of the analysis. It appears in the laws of motion and in the force laws.

Such analysis done over a number of systems should conform to two requirements: (a) It should yield a result which agrees with the experimental observations and (b)force laws should be simple in form. The formalism of Newton satisfy both these conditions.

Newton's laws of motion are three physical laws that, together, laid the foundation for classical mechanics. They describe the relationship between a body and the forces acting upon it, and its motion in response to those forces. More precisely, the first law defines the force qualitatively, the second law offers a quantitative measure of the force, and the third asserts that a single isolated force doesn't exist. These three laws have been expressed in several ways, over nearly three centuries, and can be summarised as follows:

First law: In an inertial frame of reference, an object either remains at rest or continues to move at a constant velocity, unless acted upon by a force.

Second law: In an inertial reference frame, the vector sum of the forces F on an object is equal to the mass m of that object multiplied by the acceleration **a** of the object: **F** = m**a**. (It is assumed here that the mass m is constant – see below.)

Third law: When one body exerts a force on a second body, the second body simultaneously exerts a force equal in magnitude and opposite in direction on the first body.

The three laws of motion were first compiled by Isaac Newton in his *Philosophiæ Naturalis Principia Mathematica* (Mathematical Principles of Natural Philosophy), first published in 1687. Newton used them to explain and investigate the motion of many physical objects and systems. For example, in the third volume of the text, Newton showed that these laws of motion, combined with his law of universal gravitation, explained Kepler's laws of planetary motion.

OVERVIEW

Newton's laws are applied to objects which are idealized as single point masses, in the sense that the size and shape of the object's body are neglected to focus on its motion more easily. This can be done when the object is small compared to the distances involved in its analysis, or the deformation and rotation of the body are of no importance. In this way, even a planet can be idealized as a particle for analysis of its orbital motion around a star.

In their original form, Newton's laws of motion are not adequate to characterize the motion of rigid bodies and deformable bodies. Leonhard Euler in 1750 introduced a generalization of Newton's laws of motion for rigid bodies called Euler's laws of motion, later applied as well for deformable bodies assumed as a continuum. If a body is represented as an assemblage of discrete particles, each governed by Newton's laws of motion, then Euler's laws can be derived from Newton's laws. Euler's laws can, however, be taken as axioms describing the laws of motion for extended bodies, independently of any particle structure.

Newton's laws hold only with respect to a certain set of frames of reference called Newtonian or inertial reference frames. Some authors interpret the first law as defining what an inertial reference frame is; from this point of view, the second law holds only when the observation is made from an inertial reference frame, and therefore the first law cannot be proved as a special case of the second. Other authors do treat the first law as a corollary of the second. The explicit concept of an inertial frame of reference was not developed until long after Newton's death.

In the given interpretation mass, acceleration, momentum, and (most importantly) force are assumed to be externally defined quantities. This is the most common, but not the only interpretation of the way one can consider the laws to be a definition of these quantities.

Newtonian mechanics has been superseded by special relativity, but it is still useful as an approximation when the speeds involved are much slower than the speed of light.

Laws

Newton's first law

The first law states that if the net force (the vector sum of all forces acting on an object) is zero, then the velocity of the object is constant. Velocity is a vector quantity which expresses both the object's speed and the direction of its motion; therefore, the statement that the object's velocity is constant is a statement that both its speed and the direction of its motion are constant.

The first law can be stated mathematically when the mass is a non-zero constant, as,

$$\sum \mathbf{F} = 0 \Leftrightarrow \frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t} = 0.$$

Consequently,

An object that is at rest will stay at rest unless a force acts upon it.

An object that is in motion will not change its velocity unless a force acts upon it.

This is known as uniform motion. An object continues to do whatever it happens to be doing unless a force is exerted upon it. If it is at rest, it continues in a state of rest (demonstrated when a tablecloth is skillfully whipped from under dishes on a tabletop and the dishes remain in their initial state of rest). If an object is moving, it continues to move without turning or changing its speed. This is evident in space probes that continuously move in outer space. Changes in motion must be imposed against the tendency of an object to retain its state of motion. In the absence of net forces, a moving object tends to move along a straight line path indefinitely.

Newton placed the first law of motion to establish frames of reference for which the other laws are applicable. The first law of motion postulates the existence of at least one frame of reference called a Newtonian or inertial reference frame, relative to which the motion of a particle not subject to forces is a straight line at a constant speed. Newton's first law is often referred to as the law of inertia. Thus, a condition necessary for the uniform motion of a particle relative to an inertial reference frame is that the total net force acting on it is zero. In this sense, the first law can be restated as:

In every material universe, the motion of a particle in a preferential reference frame Φ is determined by the action of forces whose total vanished for all times when and only when the velocity of the particle is constant in Φ. That is, a particle initially at rest or in uniform motion in the preferential frame Φ continues in that state unless compelled by forces to change it.

Newton's first and second laws are valid only in an inertial reference frame. Any reference frame that is in uniform motion with respect to an inertial frame is also an inertial frame, i.e. Galilean invariance or the principle of Newtonian relativity.

NEWTON'S SECOND LAW

The second law states that the rate of change of momentum of a body is directly proportional to the force applied, and this change in momentum takes place in the direction of the applied force.

$$\mathbf{F} = \frac{\mathrm{d}\mathbf{p}}{\mathrm{d}t} = \frac{\mathrm{d}(m\mathbf{v})}{\mathrm{d}t}.$$

The second law can also be stated in terms of an object's acceleration. Since Newton's second law is valid only for constant-mass systems, m can be taken outside the differentiation operator by the constant factor rule in differentiation. Thus,

$$\mathbf{F} = m\,\frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t} = m\mathbf{a},$$

where F is the net force applied, m is the mass of the body, and a is the body's acceleration. Thus, the net force applied to a body produces a proportional acceleration. In other words, if a body is accelerating, then there is a force on it. An application of this notation is the derivation of g_c.

Consistent with the first law, the time derivative of the momentum is non-zero when the momentum changes direction, even if there is no change in its magnitude; such is the case with uniform circular motion. The relationship also implies the conservation of momentum: when the net force on the body is zero, the momentum of the body is constant. Any net force is equal to the rate of change of the momentum.

Any mass that is gained or lost by the system will cause a change in momentum that is not the result of an external force. A different equation is necessary for variable-mass systems (see below).

Newton's second law is an approximation that is increasingly worse at high speeds because of relativistic effects.

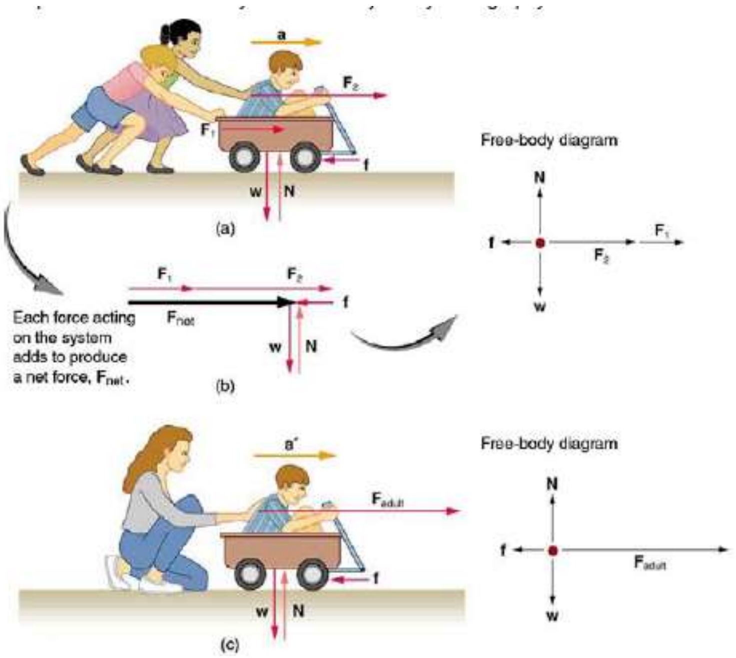


Fig 2.11: Different forces exerted on the same mass produce different accelerations. (a) Two children push a wagon with a child in it. Arrows representing all external forces are shown. The system of interest is the wagon and its rider. The weight w of the system and the support of the ground N are also shown for completeness and are assumed to cancel. The vector f represents the friction acting on the wagon, and it acts to the left, opposing the motion of the wagon. (b) All of the external forces acting on the system

add together to produce a net force, F_{net}. The free-body diagram shows all of the forces acting on the system of interest. The dot represents the center of mass of the system. Each force vector extends from this dot. Because there are two forces acting to the right, we draw the vectors collinearly. (c) A larger net external force produces a larger acceleration ($a' > a$) when an adult pushes the child.

Figure shown above is our first example of a free-body diagram, which is a technique used to illustrate all the external forces acting on a body. The body is represented by a single isolated point (or free body), and only those forces acting on the body from the outside (external forces) are shown. (These forces are the only ones shown, because only external forces acting on the body affect its motion. We can ignore any internal forces within the body.) Free-body diagrams are very useful in analyzing forces acting on a system and are employed extensively in the study and application of Newton’s laws of motion.

IMPULSE

An impulse J occurs when a force F acts over an interval of time Δt, and it is given by

$$\mathbf{J} = \int_{\Delta t} \mathbf{F}\,\mathrm{d}t.$$

Since force is the time derivative of momentum, it follows that

$$\mathbf{J} = \Delta\mathbf{p} = m\Delta\mathbf{v}.$$

This relation between impulse and momentum is closer to Newton's wording of the second law.

Impulse is a concept frequently used in the analysis of collisions and impacts.

VARIABLE-MASS SYSTEMS

Let me first tell you about a concept called Galilean Invariance, which I will be using later:

Galilean invariance or Galilean relativity states that the laws of motion are the same in all inertial frames. Galileo Galilei first described this principle in 1632 in his *Dialogue Concerning the Two Chief World Systems* using the example of a ship travelling at constant velocity, without rocking, on a smooth sea; any observer below the deck would not be able to tell whether the ship was moving or stationary.

Let me now come to variable mass system. A rocket which you use in Diwali, has a mass of say 50 g of which there is 30g of explosive and 20 g of cover, stick, etc. When fired it burns the explosive (fuel) and its mass keeps on decreasing from initial value of 50g till it reaches value of 20 g.

Variable-mass systems, like a rocket burning fuel and ejecting spent gases, are not closed and cannot be directly treated by making mass a function of time in the second law; that is, the following formula is wrong:

$$\mathbf{F}_{\mathrm{net}} = \frac{\mathrm{d}}{\mathrm{d}t}\left[m(t)\mathbf{v}(t)\right] = m(t)\frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t} + \mathbf{v}(t)\frac{\mathrm{d}m}{\mathrm{d}t}.$$

The falsehood of this formula can be seen by noting that it does not respect Galilean invariance: a variable-mass object with F = 0 in one frame will be seen to have F ≠ 0 in another frame. The correct equation of motion for a body whose mass m varies with time by either ejecting or accreting mass is obtained by applying the second law to the entire, constant-mass system consisting of the body and its ejected/accreted mass; the result is

$$\mathbf{F} + \mathbf{u}\frac{\mathrm{d}m}{\mathrm{d}t} = m\frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t}$$

where u is the velocity of the escaping or incoming mass relative to the body. From this equation one can derive the equation of motion for a varying mass system, for example, the Tsiolkovsky rocket equation. Under some conventions, the quantity u dm/dt on the left-hand side, which represents the advection of momentum, is defined as a force (the force exerted on the body by the changing mass, such as rocket exhaust) and is included in the quantity F. Then, by substituting the definition of acceleration, the equation becomes F = ma.

Solved Problems

(1) What Acceleration Can a Person Produce when Pushing a Lawn Mower?

Suppose that the net external force (push minus friction) exerted on a lawn mower is 51 N (about 11 lb) parallel to the ground. The mass of the mower is 24 kg. What is its acceleration?

Strategy

Since F_{net} and m are given, the acceleration can be calculated directly from Newton's second law as stated in $F_{net} = ma$.

Solution

The magnitude of the acceleration **a** is $= \frac{\mathrm{Fnet}}{m}$. Entering known values, give

Fig 02-12: The net force on a lawn mower is 51 N to the right. At what rate does the lawn mower accelerate to the right?

$$a = \frac{51\ \text{N}}{24\ \text{kg}}$$

Substituting the units kg · m/s^2 for N yields

$$a = \frac{51\ \text{kg} \cdot \text{m/s}^2}{24\ \text{kg}} = 2.1\ \text{m/s}^2.$$

Discussion

The direction of the acceleration is the same direction as that of the net force, which is parallel to the ground. There is no information given in this example about the individual external forces acting on the system, but we can say something about their relative magnitudes. For example, the force exerted by the person pushing the mower must be greater than the friction opposing the motion (since we know the mower moves forward), and the vertical forces must cancel if there is to be no acceleration in the vertical direction (the mower is moving only horizontally). The acceleration found is small enough to be reasonable for a person pushing a mower. Such an effort would not last too long because the person's top speed would soon be reached.

(2) What Rocket Thrust Accelerates This Sled?

Prior to manned space flights, rocket sleds were used to test aircraft, missile equipment, and physiological effects on human subjects at high speeds. They consisted of a platform that was mounted on one or two rails and propelled by several rockets. Calculate the magnitude of force exerted by each rocket, called its thrust T, for the four-rocket propulsion system shown in Figure below. The sled's initial acceleration is 49 m/s^2, the mass of the system is 2100 kg, and the force of friction opposing the motion is known to be 650 N.

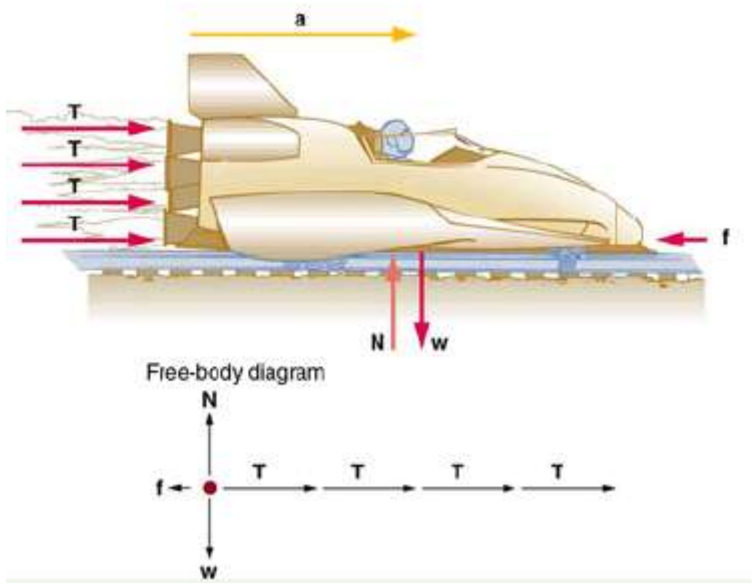


Fig 02.13: A sled experiences a rocket thrust that accelerates it to the right. Each rocket creates an identical thrust T. As in other situations where there is only horizontal acceleration, the vertical forces cancel. The ground exerts an upward force N on the system that is equal in magnitude and opposite in direction to its weight, w. The system here is the sled, its rockets, and rider, so none of the forces between these objects are considered. The arrow representing friction (f) is drawn larger than scale.

Strategy

Although there are forces acting vertically and horizontally, we assume the vertical forces cancel since there is no vertical acceleration. This leaves us with only horizontal forces and a simpler one-dimensional problem. Directions are indicated with plus or minus signs, with right taken as the positive direction. See the free-body diagram in the figure.

Solution

Since acceleration, mass, and the force of friction are given, we start with Newton's second law and look for ways to find the thrust of the engines. Since we have defined the direction of the force and acceleration as acting "to the right," we need to consider only the magnitudes of these quantities in the calculations. Hence we begin with $F_{net} = ma$, where F_{net} is the net force along the horizontal direction. We can see from adjacent Figure that the engine thrusts add, while friction opposes the thrust. In equation form, the net external force is

$$F_{net} = 4T - f \, .$$

Substituting this into Newton's second law gives

$F_{net} = ma = 4T - f$.

Substituting known values yields

$4T = ma + f = (2100 \text{ kg})(49 \text{ m/s}^2) + 650 \text{ N}$.

So the total thrust is

$4T = 1.0 \times 10^5$ N,

and the individual thrusts are

$$T = \frac{1.0 \times 10^5 \text{ N}}{4} = 2.6 \times 10^4 \text{ N}.$$

Discussion

The numbers are quite large, so the result might surprise you. Experiments such as this were performed in the early 1960s to test the limits of human endurance and the setup designed to protect human subjects in jet fighter emergency ejections.

Speeds of 1000 km/h were obtained, with accelerations of 45 g 's. (Recall that g , the acceleration due to gravity, is 9.80 m/s^2 . When we say that an acceleration is 45 g 's, it is $45 \times 9.80 \text{ m/s}^2$, which is approximately 440 m/s^2.) While living subjects are not used any more, land speeds of 10,000 km/h have been obtained with rocket sleds. In this example, as in the preceding one, the system of interest is obvious. We will see in later examples that choosing the system of interest is crucial—and the choice is not always obvious.

Newton’s second law of motion is more than a definition; it is a relationship among acceleration, force, and mass. It can help us make predictions. Each of those physical quantities can be defined independently, so the second law tells us something basic and universal about nature. The next section introduces the third and final law of motion.

NEWTON'S THIRD LAW

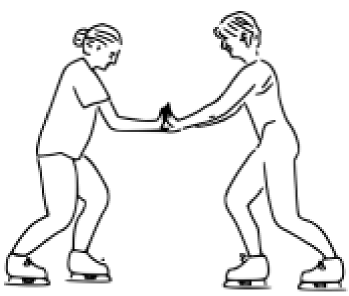


Fig 2.14: An illustration of Newton's third law in which two skaters push against each other. The first skater on the left exerts a normal force N_{12} on the second skater directed

towards the right, and the second skater exerts a normal force N_{21} on the first skater directed towards the left.

The magnitudes of both forces are equal, but they have opposite directions, as dictated by Newton's third law.

The third law states that all forces between two objects exist in equal magnitude and opposite direction: if one object A exerts a force F_A on a second object B, then B simultaneously exerts a force F_B on A, and the two forces are equal in magnitude and opposite in direction: $F_A = -F_B$. The third law means that all forces are interactions between different bodies, or different regions within one body, and thus that there is no such thing as a force that is not accompanied by an equal and opposite force. In some situations, the magnitude and direction of the forces are determined entirely by one of the two bodies, say Body A; the force exerted by Body A on Body B is called the "action", and the force exerted by Body B on Body A is called the "reaction". This law is sometimes referred to as the action-reaction law, with F_A called the "action" and F_B the "reaction". In other situations the magnitude and directions of the forces are determined jointly by both bodies and it isn't necessary to identify one force as the "action" and the other as the "reaction". The action and the reaction are simultaneous, and it does not matter which is called the action and which is called reaction; both forces are part of a single interaction, and neither force exists without the other.

The two forces in Newton's third law are of the same type (e.g., if the road exerts a forward frictional force on an accelerating car's tyres, then it is also a frictional force that Newton's third law predicts for the tires pushing backward on the road).
From a conceptual standpoint, Newton's third law is seen when a person walks: they push against the floor, and the floor pushes against the person. Similarly, the tyres of a car push against the road while the road pushes back on the tyres —the tyres and road simultaneously push against each other. In swimming, a person interacts with the water, pushing the water backward, while the water simultaneously pushes the person forward—both the person and the water push against each other. The reaction forces account for the motion in these examples. These forces depend on friction; a person or car on ice, for example, may be unable to exert the action force to produce the needed reaction force.

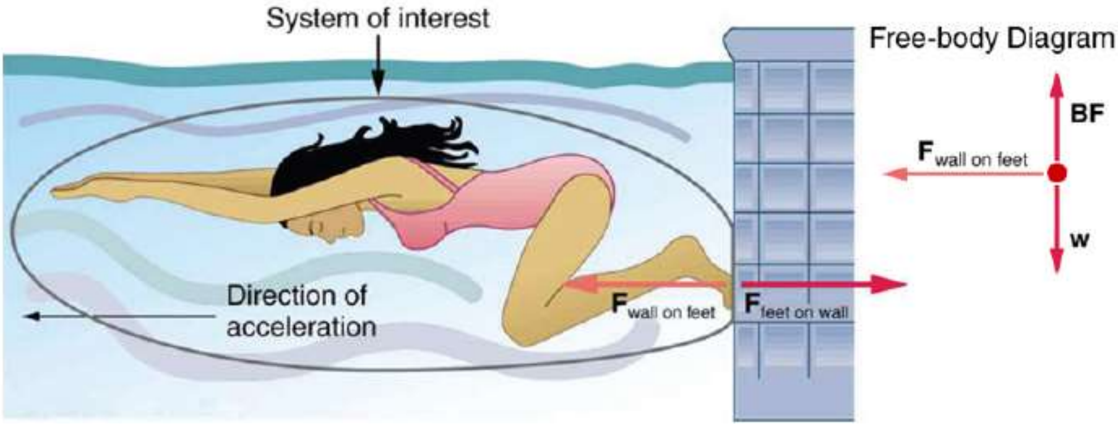


Fig 02-15: When the swimmer exerts a force $F_{\text{feet on wall}}$ on the wall, she accelerates in the direction opposite to that of her push. This means the net external force on her is in the direction opposite to $F_{\text{feet on wall}}$. This opposition occurs because, in accordance with Newton's third law of motion, the wall exerts a force $F_{\text{wall on feet}}$ on her, equal in magnitude but in the direction opposite to the one she exerts on it. The line around the swimmer indicates the system of interest. Note that $F_{\text{feet on wall}}$ does not act on this system (the swimmer) and, thus, does not cancel $F_{\text{wall on feet}}$. Thus the free-body diagram shows only $F_{\text{wall on feet}}$, w, the gravitational force, and BF, the buoyant force of the water supporting the swimmer's weight. The vertical forces w and BF cancel since there is no vertical motion.

We can readily see Newton's third law at work by taking a look at how people move about. Consider a swimmer pushing off from the side of a pool, as illustrated in Figure above. She pushes against the pool wall with her feet and accelerates in the direction opposite to that of her push. The wall has exerted an equal and opposite force back on the swimmer. You might think that two equal and opposite forces would cancel, but they do not because they act on different systems. In this case, there are two systems that we could investigate: the swimmer or the wall. If we select the swimmer to be the system of interest, as in the figure, then $F_{\text{wall on feet}}$ is an external force on this system and affects its motion. The swimmer moves in the direction of $F_{\text{wall on feet}}$. In contrast, the force $F_{\text{wall on wall}}$ acts on the wall and not on our system of interest. Thus $F_{\text{feet on wall}}$ does not directly affect the motion of the system and does not cancel $F_{\text{wall on feet}}$. Note that the swimmer pushes in the direction opposite to that in which she wishes to move. The reaction to her push is thus in the desired direction.

Other examples of Newton's third law are easy to find. As a professor paces in front of a whiteboard, she exerts a force backward on the floor. The floor exerts a reaction force forward on the professor that causes her to accelerate forward. Similarly, a car accelerates because the ground pushes forward on the drive wheels in reaction to the drive wheels pushing backward on the ground. You can see evidence of the wheels pushing backward when tyres spin on a gravel road and throw rocks backward.

In another example, rockets move forward by expelling gas backward at high velocity. This means the rocket exerts a large backward force on the gas in the rocket combustion chamber, and the gas therefore exerts a large reaction force forward on the rocket. This reaction force is called thrust. It is a common misconception that rockets propel themselves by pushing on the ground or on the air behind them. They actually work better in a vacuum, where they can more readily expel the exhaust gases.

Helicopters similarly create lift by pushing air down, thereby experiencing an upward reaction force. Birds and airplanes also fly by exerting force on air in a direction opposite to that of whatever force they need. For example, the wings of a bird force air downward and backward in order to get lift and move forward. An octopus propels itself in the water by ejecting water through a funnel from its body, similar to a jet ski. In a situation similar to Sancho's, professional cage fighters experience reaction forces when they punch, sometimes breaking their hand by hitting an opponent's body.

Don't get confused between Newton's third law and law of addition of forces. The Sun exerts a gravitational pull on earth which is same in magnitude by opposite in direction to that exerted by the earth on the sun. It does not mean that the net pull is zero, hence no gravity. The core idea is that the points of application of the action and reaction forces are not same.

Solved Problems

Example 01: Getting Up To Speed: Choosing the Correct System

A physics professor pushes a cart of demonstration equipment to a lecture hall, as seen in Figure 2-16. Her mass is 65.0 kg, the cart's is 12.0 kg, and the equipment's is 7.0 kg. Calculate the acceleration produced when the professor exerts a backward force of 150 N on the floor. All forces opposing the motion, such as friction on the cart's wheels and air resistance, total 24.0 N.

Strategy

Since they accelerate as a unit, we define the system to be the professor, cart, and equipment. This is System 1 in Figure. The professor pushes backward with a force F_{foot} of 150 N. According to Newton's third law, the floor exerts a forward reaction force F_{floor} of 150 N on System 1. Because all motion is horizontal, we can assume there is no net force in the vertical direction. The problem is therefore one-dimensional along the horizontal direction. As noted, f opposes the motion and is thus in the opposite

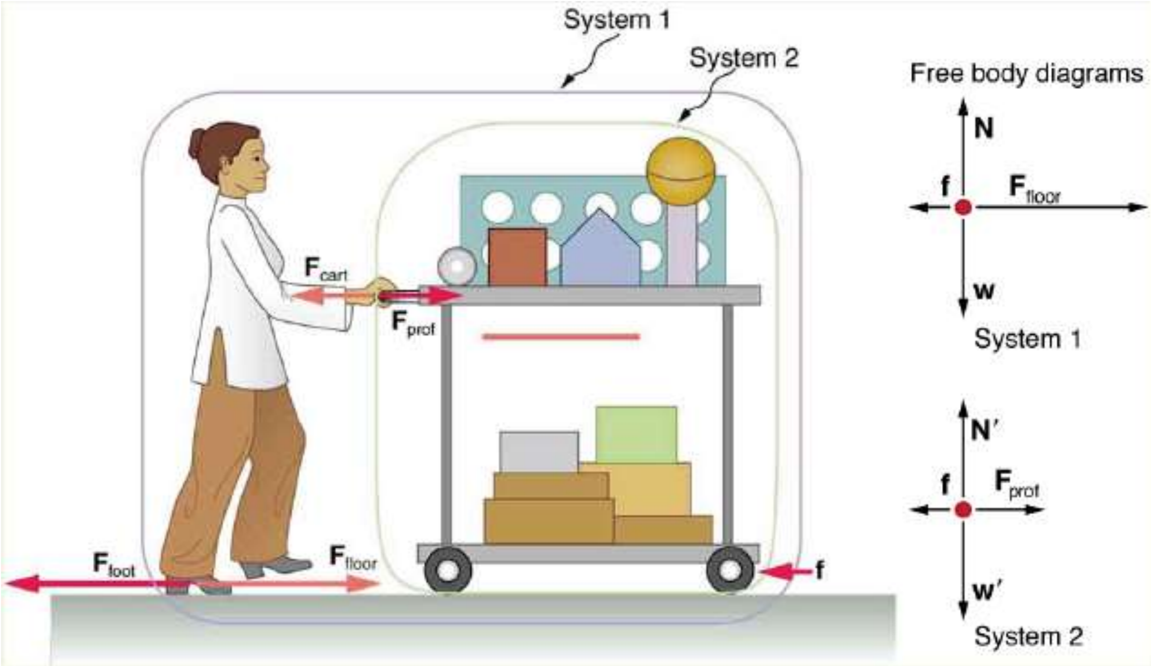


Fig 02.16: A professor pushes a cart of demonstration equipment. The lengths of the arrows are proportional to the magnitudes of the forces (except for f, since it is too small to draw to scale). Different questions are asked in each example; thus, the system of interest must be defined differently for each. System 1 is appropriate for Example 4.4, since it asks for the acceleration of the entire group of objects. Only F_{floor} and f are external forces acting on System 1 along the line of motion. All other forces either cancel or act on the outside world. System 2 is chosen for this example so that F_{prof} will be an external force and enter into Newton's second law. Note that the free-body diagrams, which allow us to apply Newton's second law, vary with the system chosen.

direction of F_{floor}. Note that we do not include the forces F_{prof} or F_{cart} because these are internal forces, and we do not include F_{foot} because it acts on the floor, not on the system. There are no other significant forces acting on System 1. If the net external force can be found from all this information, we can use Newton's second law to find the acceleration as requested. See the free-body diagram in the figure.

Solution

Newton's second law is given by

$$a = \frac{F_{net}}{m}.$$

The net external force on System 1 is deduced from the Figure and the discussion above to be

$$F_{net} = F_{floor} - f = 150\ \text{N} - 24.0\ \text{N} = 126\ \text{N}.$$

The mass of System 1 is

$$m = (65.0 + 12.0 + 7.0)\ \text{kg} = 84\ \text{kg}.$$

These values of F_{net} and m produce an acceleration of

$$a = \frac{F_{net}}{m},$$
$$a = \frac{126\ \text{N}}{84\ \text{kg}} = 1.5\ \text{m/s}^2.$$

Discussion

None of the forces between components of System 1, such as between the professor's hands and the cart, contribute to the net external force because they are internal to System 1. Another way to look at this is to note that forces between components of a system cancel because they are equal in magnitude and opposite in direction. For example, the force exerted by the professor on the cart, results in an equal and opposite force back on her. In this case both forces act on the same system and, therefore, cancel. Thus internal forces (between components of a system) cancel. Choosing System 1 was crucial to solving this problem.

EXAMPLE (2) FORCE ON THE CART—CHOOSING A NEW SYSTEM

Calculate the force the professor exerts on the cart using data from the previous example if needed.

Strategy

If we now define the system of interest to be the cart plus equipment (System 2 in Figure for previous example), then the net external force on System 2 is the force the professor exerts on the cart minus friction. The force she exerts on the cart, F_{prof} , is an external force acting on System 2. F_{prof} was internal to System 1, but it is external to System 2 and will enter Newton's second law for System 2.

Solution

Newton's second law can be used to find F_{prof} . Starting with

$$a = \frac{F_{net}}{m}$$

and noting that the magnitude of the net external force on System 2 is

$$F_{net} = F_{prof} - f,$$

we solve for F_{prof} , the desired quantity:

$$F_{prof} = F_{net} + f.$$

The value of f is given, so we must calculate F_{net}. That can be done since both the acceleration and mass of System 2 are known. Using Newton's second law we see that

$$F_{net} = ma,$$

where the mass of System 2 is 19.0 kg (m = 12.0 kg + 7.0 kg) and its acceleration was found to be a = 1.5 m/s^2 in the previous example. Thus,

$$F_{net} = ma,$$
$$F_{net} = (19.0\ \text{kg})(1.5\ \text{m/s}^2) = 29\ \text{N}.$$

Now we can find the desired force:

$$F_{prof} = F_{net} + f,$$
$$F_{prof} = 29\ \text{N} + 24.0\ \text{N} = 53\ \text{N}.$$

Discussion

It is interesting that this force is significantly less than the 150-N force the professor exerted backward on the floor. Not all of that 150-N force is transmitted to the cart; some of it accelerates the professor.

The choice of a system is an important analytical step both in solving problems and in thoroughly understanding the physics of the situation (which is not necessarily the same thing).

History

Newton's 1st Law

From the original Latin of Newton's Principia:

“ Lex I: Corpus omne perseverare in statu suo quiescendi vel movendi uniformiter in directum, nisi quatenus a viribus impressis cogitur statum illum mutare. ”

Translated to English, this reads:

“ Law I: Every body persists in its state of being at rest or of moving uniformly straight forward, except in so far as it is compelled to change its state by force impressed. ”

AXIOMATA
SIVE
LEGES MOTUS

Lex. I.

Lex. II.

Fig 2.17: Newton's First and second laws, in Latin, from the original 1687 Principia Mathematica

The ancient Greek philosopher Aristotle had the view that all objects have a natural place in the universe: that heavy objects (such as rocks) wanted to be at rest on the Earth and that light objects like smoke wanted to be at rest in the sky and the stars wanted to remain in the heavens. He thought that a body was in its natural state when it was at rest, and for the body to move in a straight line at a constant speed an external agent was needed continually to propel it, otherwise it would stop moving. Galileo Galilei, however, realised that a force is necessary to change the velocity of a body, i.e., acceleration, but no force is needed to maintain its velocity. In other words, Galileo stated that, in the absence of a force, a moving object will continue moving. (The tendency of objects to resist changes in motion was what Johannes Kepler had called inertia.) This insight was refined by Newton, who made it into his first law, also known as the "law of inertia"—no force means no acceleration, and hence the body will maintain its velocity. As Newton's first law is a restatement of the law of inertia which Galileo had already described, Newton appropriately gave credit to Galileo.

The law of inertia apparently occurred to several different natural philosophers and scientists independently, including Thomas Hobbes in his Leviathan. The 17th century philosopher and mathematician René Descartes also formulated the law, although he did not perform any experiments to confirm it.

Newton's 2nd Law

Newton's original Latin reads:

" Lex II: Mutationem motus proportionalem esse vi motrici impressae, et fieri secundum lineam rectam qua vis illa imprimitur. "

This was translated quite closely in Motte's 1729 translation as:

" Law II: The alteration of motion is ever proportional to the motive force impress'd; and is made in the direction of the right line in which that force is impress'd."

According to modern ideas of how Newton was using his terminology, this is understood, in modern terms, as an equivalent of:

The change of momentum of a body is proportional to the impulse impressed on the body, and happens along the straight line on which that impulse is impressed.

This may be expressed by the formula F = p', where p' is the time derivative of the momentum p. This equation can be seen clearly in the Wren Library of Trinity College, Cambridge, in a glass case in which Newton's manuscript is open to the relevant page. Motte's 1729 translation of Newton's Latin continued with Newton's commentary on the second law of motion, reading:

If a force generates a motion, a double force will generate double the motion, a triple force triple the motion, whether that force be impressed altogether and at once, or gradually and successively. And this motion (being always directed the same way with the generating force), if the body moved before, is added to or subtracted from the former motion, according as they directly conspire with or are directly contrary to each other; or obliquely joined, when they are oblique, so as to produce a new motion compounded from the determination of both.

The sense or senses in which Newton used his terminology, and how he understood the second law and intended it to be understood, have been extensively discussed by historians of science, along with the relations between Newton's formulation and modern formulations.

Newton's 3rd Law

" Lex III: Actioni contrariam semper et æqualem esse reactionem: sive corporum duorum actiones in se mutuo semper esse æquales et in partes contrarias dirigi. "

Translated to English, this reads:

" Law III: To every action there is always opposed an equal reaction: or the mutual actions of two bodies upon each other are always equal, and directed to contrary parts. "

Newton's Scholium (explanatory comment) to this law:

Whatever draws or presses another is as much drawn or pressed by that other. If you press a stone with your finger, the finger is also pressed by the stone. If a horse draws a stone tied to a rope, the horse (if I may so say) will be equally drawn back towards the stone: for the distended rope, by the same endeavour to relax or unbend itself, will draw the horse as much towards the stone, as it does the stone towards the horse, and will obstruct the progress of the one as much as it advances that of the other. If a body impinges upon another, and by its force changes the motion of the other, that body also (because of the equality of the mutual pressure) will undergo an equal change, in its own motion, toward the contrary part. The changes made by these actions are equal, not in the velocities but in the motions of the bodies; that is to say, if the bodies are not hindered by any other impediments. For, as the motions are equally changed, the changes of the velocities made toward contrary parts are reciprocally proportional to the bodies. This law takes place also in attractions, as will be proved in the next scholium.

In the above, as usual, motion is Newton's name for momentum, hence his careful distinction between motion and velocity.

Newton used the third law to derive the law of conservation of momentum; from a deeper perspective, however, conservation of momentum is the more fundamental idea (derived via Noether's theorem from Galilean invariance), and holds in cases where Newton's third law appears to fail, for instance when force fields as well as particles carry momentum, and in quantum mechanics.

IMPORTANCE AND RANGE OF VALIDITY

Newton's laws were verified by experiment and observation for over 200 years, and they are excellent approximations at the scales and speeds of everyday life. Newton's laws of motion, together with his law of universal gravitation and the mathematical techniques of calculus, provided for the first time a unified quantitative explanation for a wide range of physical phenomena.

These three laws hold to a good approximation for macroscopic objects under everyday conditions. However, Newton's laws (combined with universal gravitation and classical electrodynamics) are inappropriate for use in certain circumstances, most notably at very small scales, very high speeds (in special relativity, the Lorentz factor must be included in the expression for momentum along with the rest mass and velocity) or very strong gravitational fields. Therefore, the laws cannot be used to explain phenomena such as conduction of electricity in a semiconductor, optical properties of substances, errors in non-relativistically corrected GPS systems and superconductivity. Explanation of these phenomena requires more sophisticated physical theories, including general relativity and quantum field theory.

In quantum mechanics, concepts such as force, momentum, and position are defined by linear operators that operate on the quantum state; at speeds that are much lower than the speed of light, Newton's laws are just as exact for these operators as they are for classical objects. At speeds comparable to the speed of light, the second law holds in the original form $F = dp/dt$, where F and p are four-vectors.

RELATIONSHIP TO THE CONSERVATION LAWS

In modern physics, the laws of conservation of momentum, energy, and angular momentum are of more general validity than Newton's laws, since they apply to both light and matter, and to both classical and non-classical physics.

This can be stated simply, "Momentum, energy and angular momentum cannot be created or destroyed."

Because force is the time derivative of momentum, the concept of force is redundant and subordinate to the conservation of momentum, and is not used in fundamental theories (e.g., quantum mechanics, quantum electrodynamics, general relativity, etc.). The standard model explains in detail how the three fundamental forces known as gauge forces originate out of exchange by virtual particles. Other forces, such as gravity and fermionic degeneracy pressure, also arise from the momentum conservation. Indeed, the conservation of 4-momentum in inertial motion via curved space-time results in what we call gravitational force in general relativity theory. The application of the space derivative (which is a momentum operator in quantum mechanics) to the overlapping wave functions of a pair of fermions (particles with half-integer spin) results in shifts of maxima of compound wavefunction away from each other, which is observable as the "repulsion" of the fermions.

Newton stated the third law within a world-view that assumed instantaneous action at a distance between material particles. However, he was prepared for philosophical criticism of this action at a distance, and it was in this context that he stated the famous phrase "I feign no hypotheses". In modern physics, action at a distance has been completely eliminated, except for subtle effects involving quantum entanglement. (In particular, this refers to Bell's theorem – that no local model can reproduce the predictions of quantum theory.) Despite only being an approximation, in modern engineering and all practical applications involving the motion of vehicles and satellites, the concept of action at a distance is used extensively.

The discovery of the second law of thermodynamics by Carnot in the 19th century showed that not every physical quantity is conserved over time, thus disproving the validity of inducing the opposite metaphysical view from Newton's laws. Hence, a "steady-state" worldview based solely on Newton's laws and the conservation laws does not take entropy into account.

Normal, Tension, and Other Examples of Forces

Forces are given many names, such as push, pull, thrust, lift, weight, friction, and tension. Traditionally, forces have been grouped into several categories and given names relating to their source, how they are transmitted, or their effects. The most important of these categories are discussed in this section, together with some interesting applications. Further examples of forces are discussed later in this text.

Normal Force

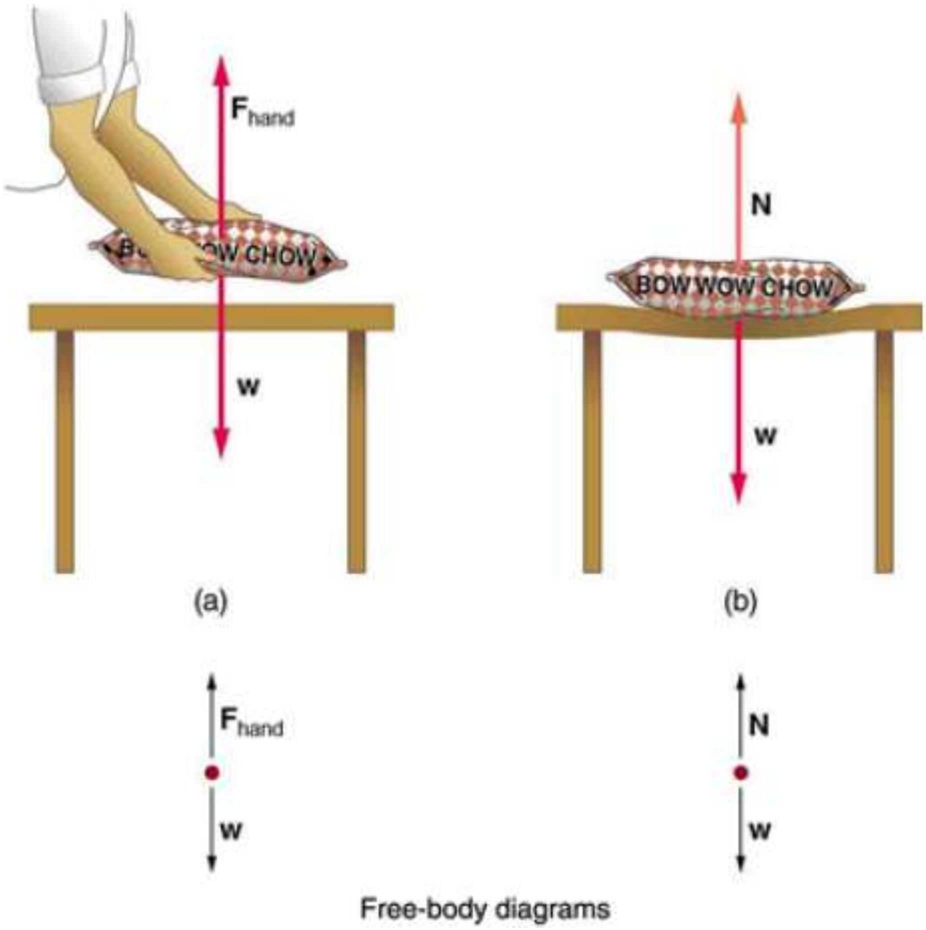


Fig 02-18: (a) The person holding the bag of dog food must supply an upward force F_{hand} equal in magnitude and opposite in direction to the weight of the food w . (b) The card table sags when the dog food is placed on it, much like a stiff trampoline. Elastic restoring forces in the table grow as it sags until they supply a force N equal in magnitude and opposite in direction to the weight of the load.

Weight (also called force of gravity) is a pervasive force that acts at all times and must be counteracted to keep an object from falling. You definitely notice that you must support the weight of a heavy object by pushing up on it when you hold it stationary, as illustrated in Figure. But how do inanimate objects like a table support the weight of a mass placed on them, such as shown in Figure (b)? When the bag of dog food is placed on the table, the table actually sags slightly under the load. This would be noticeable if the load was placed on a card table, but even rigid objects deform when a force is applied to them.

Unless the object is deformed beyond its limit, it will exert a restoring force much like a deformed spring (or trampoline or diving board). The greater the deformation, the greater the restoring force. So when the load is placed on the table, the table sags until the restoring force becomes as large as the weight of the load. At this point the net external force on the load is zero. That is the situation when the load is stationary on the table. The table sags quickly, and the sag is slight so we do not notice it. But it is similar to the sagging of a trampoline when you climb onto it.

We must conclude that whatever supports a load, be it animate or not, must supply an upward force equal to the weight of the load, as we assumed in a few of the previous examples. If the force supporting a load is perpendicular to the surface of contact between the load and its support, this force is defined to be a normal force and here is given the symbol N. (This is not the unit for force N.) The word normal means perpendicular to a surface. The normal force can be less than the object's weight if the object is on an incline, as you will see in the next example.

Common Misconception: Normal Force (N) vs. Newton (N)

In this section we have introduced the quantity normal force, which is represented by the variable $\mathbf{N}$. This should not be confused with the symbol for the newton, which is also represented by the letter N. These symbols are particularly important to distinguish because the units of a normal force ($\mathbf{N}$) happen to be newtons (N). For example, the normal force $\mathbf{N}$ that the floor exerts on a chair might be $\mathbf{N} = 100\ \mathrm{N}$. One important difference is that normal force is a vector, while the newton is simply a unit. Be careful not to confuse these letters in your calculations! You will encounter more similarities among variables and units as you proceed in physics. Another example of this is the quantity work (W) and the unit watts (W).

Solved Problems

Example (1) Weight on an Inclined Plane, a Two-Dimensional Problem

Consider the skier on a slope shown in Figure. Her mass including equipment is 60.0 kg. (a) What is her acceleration if friction is negligible? (b) What is her acceleration if friction is known to be 45.0 N?

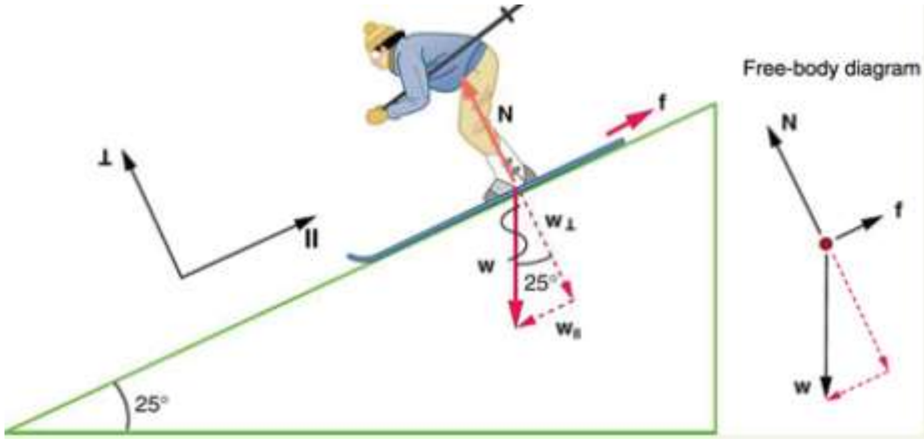


Fig 02-19: Since motion and friction are parallel to the slope, it is most convenient to project all forces onto a coordinate system where one axis is parallel to the slope and the other is perpendicular (axes shown to left of skier). N is perpendicular to the slope and f is parallel to the slope, but w has components along both axes, namely w⊥ and w ∥. N is equal in magnitude to w⊥, so that there is no motion perpendicular to the slope, but f is less than w ∥, so that there is an acceleration down the slope (along the parallel axis).

Strategy

This is a two-dimensional problem, since the forces on the skier (the system of interest) are not parallel. The approach we have used in two-dimensional kinematics also works very well here. Choose a convenient coordinate system and project the vectors onto its axes, creating two connected one-dimensional problems to solve. The most convenient coordinate system for motion on an incline is one that has one coordinate parallel to the slope and one perpendicular to the slope. (Remember that motions along mutually perpendicular axes are independent.) We use the symbols ⊥ and ∥ to represent perpendicular and parallel, respectively. This choice of axes simplifies this type of problem, because there is no motion perpendicular to the slope and because friction is always parallel to the surface between two objects. The only external forces acting on the system are the skier's weight, friction, and the support of the slope, respectively labeled w, f , and N in Figure 02-19. N is always perpendicular to the slope, and f is parallel to it. But w is not in the direction of either axis, and so the first step we take is to project it into components along the chosen axes, defining w ∥ to be the component of weight parallel to the slope and w⊥ the component of weight perpendicular to the slope. Once this is done, we can consider the two separate problems of forces parallel to the slope and forces perpendicular to the slope.

Solution

The magnitude of the component of the weight parallel to the slope is $w_{\parallel} = w \sin(25^\circ) = mg \sin(25^\circ)$, and the magnitude of the component of the weight perpendicular to the slope is $w_{\perp} = w \cos(25^\circ) = mg \cos(25^\circ)$.

(a) Neglecting friction. Since the acceleration is parallel to the slope, we need to consider only the forces parallel to the slope. (Forces perpendicular to the slope add to zero, since there is no acceleration in that direction.) The forces parallel to the slope are the amount of the skier's weight parallel to the slope $w_{\parallel}$ and friction f. Using Newton's second law, with subscripts to denote quantities parallel to the slope,

$$a_{\parallel} = \frac{F_{\text{net}\parallel}}{m}$$

where $F_{\text{net}\parallel} = w_{\parallel} = mg \sin(25^\circ)$, assuming no friction for this part, so that

$$a_{\parallel} = \frac{F_{\text{net}\parallel}}{m} = \frac{mg \sin(25^\circ)}{m} = g \sin(25^\circ)$$
$$(9.80 \text{ m/s}^2)(0.4226) = 4.14 \text{ m/s}^2$$

is the acceleration.

(b) Including friction. We now have a given value for friction, and we know its direction is parallel to the slope and it opposes motion between surfaces in contact. So the net external force is now $F_{\text{net}\parallel} = w_{\parallel} - f$,

and substituting this into Newton's second law, $a_{\parallel} = (F_{\text{net}\parallel})/m$, gives

$$a_{\parallel} = \frac{F_{\text{net}\parallel}}{m} = \frac{w_{\parallel} - f}{m} = \frac{mg \sin(25^\circ) - f}{m}.$$

We substitute known values to obtain

$$a_{\parallel} = \frac{(60.0 \text{ kg})(9.80 \text{ m/s}^2)(0.4226) - 45.0 \text{ N}}{60.0 \text{ kg}},$$

which yields

$a_{\parallel} = 3.39 \text{ m/s}^2$,

which is the acceleration parallel to the incline when there is 45.0 N of opposing friction.

Discussion

Since friction always opposes motion between surfaces, the acceleration is smaller when there is friction than when there is none. In fact, it is a general result that if friction on an incline is negligible, then the acceleration down the incline is $a = g \sin\theta$, regardless of mass. This is related to the previously discussed fact that all objects fall with the same acceleration in the absence of air resistance. Similarly, all objects, regardless of mass, slide down a frictionless incline with the same acceleration (if the angle is the same).

Tension

A tension is a force along the length of a medium, especially a force carried by a flexible medium, such as a rope or cable. The word "tension" comes from a Latin word meaning "to stretch." Not coincidentally, the flexible cords that carry muscle forces to other parts of the body are called tendons. Any flexible connector, such as a string, rope, chain, wire, or cable, can exert pulls only parallel to its length; thus, a force carried by a flexible connector is a tension with direction parallel to the connector. It is important to understand that tension is a pull in a connector. In contrast, consider the phrase: "You can't push a rope." The tension force pulls outward along the two ends of a rope.

Consider a person holding a mass on a rope as shown in Figure 02-20.

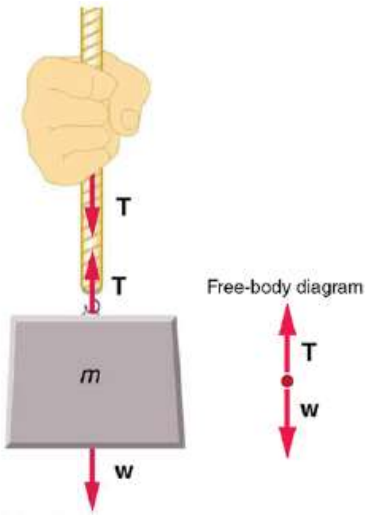


Fig 02-20: When a perfectly flexible connector (one requiring no force to bend it) such as this rope transmits a force T, that force must be parallel to the length of the rope, as shown. The pull such a flexible connector exerts is a tension. Note that the rope pulls with equal force but in opposite directions on the hand and the supported mass (neglecting the weight of the rope). This is an example of Newton's third law. The rope is the medium that carries the equal and opposite forces between the two objects. The

tension anywhere in the rope between the hand and the mass is equal. Once you have determined the tension in one location, you have determined the tension at all locations along the rope.

Tension in the rope must equal the weight of the supported mass, as we can prove using Newton's second law. If the 5.00-kg mass in the figure is stationary, then its acceleration is zero, and thus $F_{net} = 0$. The only external forces acting on the mass are

it's weight w and the tension T supplied by the rope. Thus,

$F_{net} = T - w = 0,$

where T and w are the magnitudes of the tension and weight and their signs indicate direction, with up being positive here.

Thus, just as you would expect, the tension equals the weight of the supported mass:

$T = w = mg.$

For a 5.00-kg mass, then (neglecting the mass of the rope) we see that

$T = mg = (5.00\ kg)(9.80\ m/s2\ ) = 49.0\ N.$

If we cut the rope and insert a spring, the spring would extend a length corresponding to a force of 49.0 N, providing a direct observation and measure of the tension force in the rope.

Flexible connectors are often used to transmit forces around corners, such as in a hospital traction system, a finger joint, or a bicycle brake cable. If there is no friction, the tension is transmitted undiminished. Only its direction change, and it is always parallel to the flexible connector. This is illustrated in Figure below.

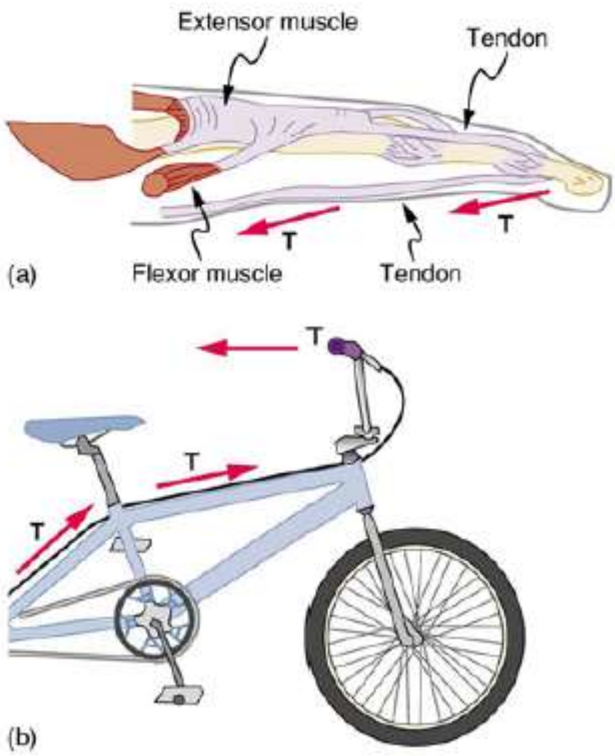


Fig 02-21: (a) Tendons in the finger carry force T from the muscles to other parts of the finger, usually changing the force's direction, but not its magnitude (the tendons are relatively friction free). (b) The brake cable on a bicycle carries the tension T from the handlebars to the brake mechanism. Again, the direction but not the magnitude of T is changed.

Solved Problems

Example (1) What Is the Tension in a Tightrope?

Calculate the tension in the wire supporting the 70.0-kg tight rope walker shown in Figure.

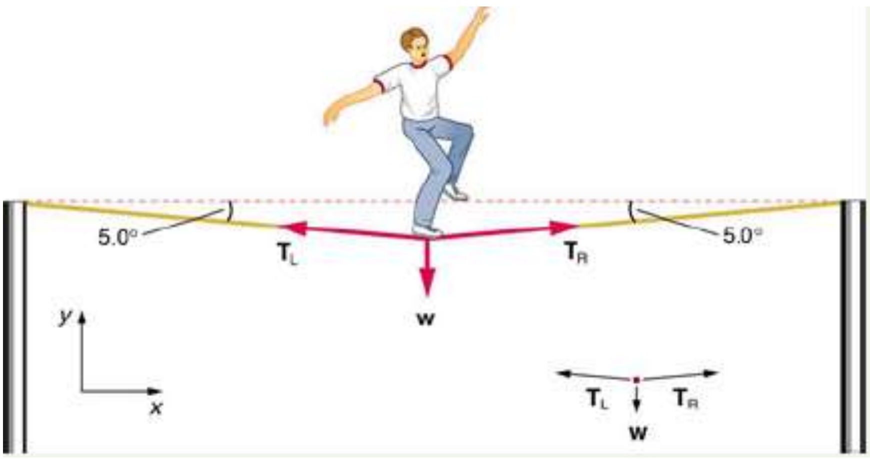


Fig 02-21: The weight of a tightrope walker causes a wire to sag by 5.0 degrees. The system of interest here is the point in the wire at which the tightrope walker is standing.

Strategy

As you can see in the figure, the wire is not perfectly horizontal (it cannot be!), but is bent under the person's weight. Thus, the tension on either side of the person has an upward component that can support his weight. As usual, forces are vectors represented pictorially by arrows having the same directions as the forces and lengths proportional to their magnitudes. The system is the tight rope walker, and the only external forces acting on him are his weight w and the two tensions TL (left tension) and TR (right tension), as illustrated. It is reasonable to neglect the weight of the wire itself. The net external force is zero since the system is stationary. A little trigonometry can now be used to find the tensions. One conclusion is possible at the outset—we can see from part (b) of the figure that the magnitudes of the tensions TL and TR must be equal. This is because there is no horizontal acceleration in the rope, and the only forces acting to the left and right are TL and TR.

Thus, the magnitude of those forces must be equal so that they cancel each other.

Whenever we have two-dimensional vector problems in which no two vectors are parallel, the easiest method of solution is to pick a convenient coordinate system and project the vectors onto its axes. In this case the best coordinate system has one axis horizontal and the other vertical. We call the horizontal as x -axis and the vertical as y -axis.

Solution

First, we need to resolve the tension vectors into their horizontal and vertical components. It helps to draw a new free-body diagram showing all of the horizontal and vertical components of each force acting on the system.

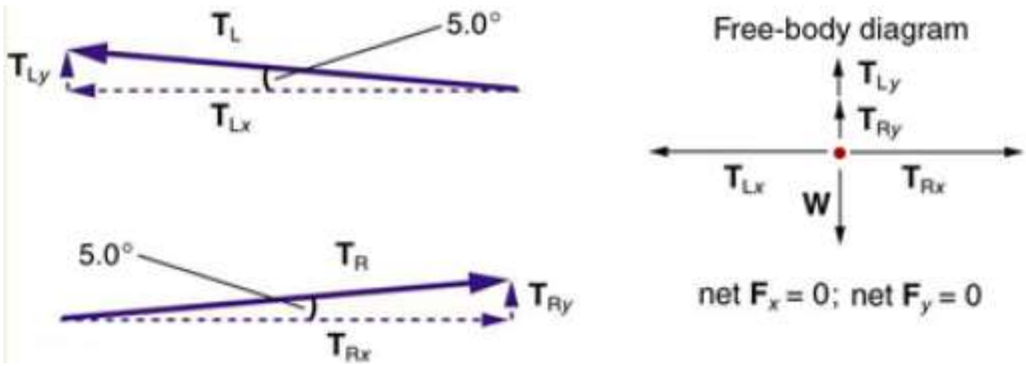


Fig 02-22: When the vectors are projected onto vertical and horizontal axes, their components along those axes must add to zero, since the tightrope walker is stationary. The small angle results in T being much greater than w.

Consider the horizontal components of the forces (denoted with a subscript x):

$F_{netx} = T_{Lx} - T_{Rx}$.

The net external horizontal force $F_{netx} = 0$, since the person is stationary. Thus,

$F_{netx} = 0 = T_{Lx} - T_{Rx}$

$T_{Lx} = T_{Rx}$.

Now, observe Figure 02-22. You can use trigonometry to determine the magnitude of T_L and T_R. Notice that:

$$\begin{aligned}
\cos(5.0^\circ) &= \frac{T_{Lx}}{T_L} \\
T_{Lx} &= T_L \cos(5.0^\circ) \\
\cos(5.0^\circ) &= \frac{T_{Rx}}{T_R} \\
T_{Rx} &= T_R \cos(5.0^\circ).
\end{aligned}$$

Equating T_{Lx} and T_{Rx} :

$T_L \cos(5.0^\circ) = T_R \cos(5.0^\circ)$.

Thus,

$T_L = T_R = T$,

as predicted. Now, considering the vertical components (denoted by a subscript y), we can solve for T . Again, since the person is stationary, Newton's second law implies that net Fy = 0 . Thus, as illustrated in the free-body diagram in Figure 02-22,

$F_{nety} = T_{Ly} + T_{Ry} - w = 0$.

Observing Figure, we can use trigonometry to determine the relationship between T_{Ly} , T_{Ry} , and T . As we determined from the analysis in the horizontal direction,

$T_L = T_R = T$:

$$\begin{aligned}
\sin(5.0^\circ) &= \frac{T_{Ly}}{T_L} \\
T_{Ly} = T_L \sin(5.0^\circ) &= T \sin(5.0^\circ) \\
\sin(5.0^\circ) &= \frac{T_{Ry}}{T_R} \\
T_{Ry} = T_R \sin(5.0^\circ) &= T \sin(5.0^\circ).
\end{aligned}$$

Now, we can substitute the values for T_{Ly} and T_{Ry} , into the net force equation in the vertical direction:

$$\begin{aligned}
F_{\text{net}y} &= T_{\text{L}y} + T_{\text{R}y} - w = 0 \\
F_{\text{net}y} &= T \sin(5.0^\circ) + T \sin(5.0^\circ) - w = 0 \\
2T \sin(5.0^\circ) - w &= 0 \\
2T \sin(5.0^\circ) &= w
\end{aligned}$$

And

$$T = \frac{w}{2 \sin(5.0^\circ)} = \frac{mg}{2 \sin(5.0^\circ)},$$

So that

$$T = \frac{(70.0\ \text{kg})(9.80\ \text{m/s}^2)}{2(0.0872)},$$

and the tension is

T = 3900 N. (4.54)

Discussion

Note that the vertical tension in the wire acts as a normal force that supports the weight of the tightrope walker. The tension is almost six times the 686-N weight of the tightrope walker. Since the wire is nearly horizontal, the vertical component of its tension is only a small fraction of the tension in the wire. The large horizontal components are in opposite directions and cancel, and so most of the tension in the wire is not used to support the weight of the tightrope walker.

Example (02) What Does the Weighing (Spring) Scale Read in an Elevator (Lift)?

Figure shows a 75.0-kg man (weight of about 165 lb) standing on a bathroom scale in an elevator. Calculate the scale reading: (a) if the elevator accelerates upward at a rate of 1.20 m/s^2 , and (b) if the elevator moves upward at a constant speed of 1 m/s.

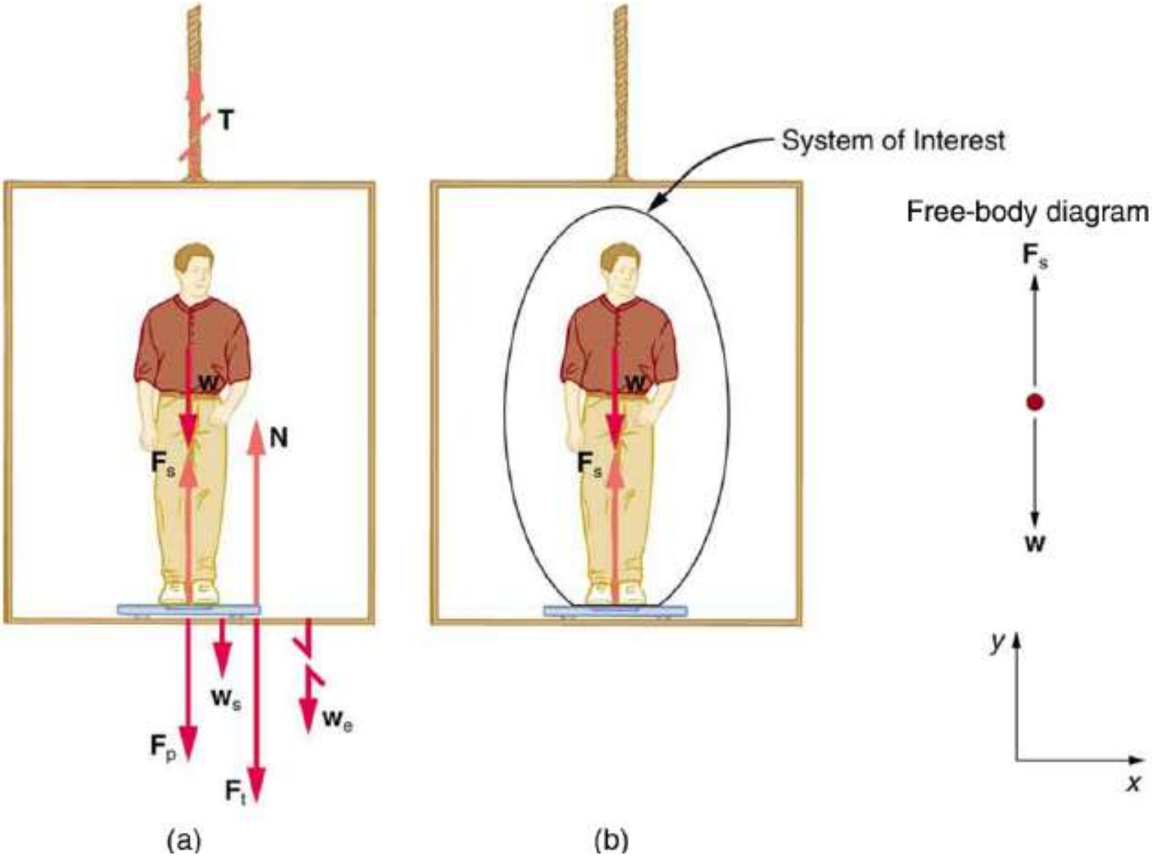


Fig 02-23: (a) Various forces acting when a person stands on a bathroom scale in an elevator. The arrows are approximately correct for when the elevator is accelerating upward—broken arrows represent forces too large to be drawn to scale. T is the tension in the supporting cable, w is the weight of the person, w_s is the weight of the scale, w_e is the weight of the elevator, F_s is the force of the scale on the person, F_p is the force of the person on the scale, F_t is the force of the scale on the floor of the elevator, and N is the force of the floor upward on the scale. (b) The free-body diagram shows only the external forces acting on the designated system of interest—the person.

Strategy

If the scale (spring balance) is accurate, its reading will equal Fp , the magnitude of the force the person exerts downward on it. Figure 02-23 shows the numerous forces acting on the elevator, scale, and person. It makes this one-dimensional problem look much more formidable than if the person is chosen to be the system of interest and a free-body diagram is drawn as in Figure (b). Analysis of the free-body diagram using Newton's laws can produce answers to both parts (a) and (b) of this example, as well as some other questions that might arise. The only forces acting on the person are his weight w and the upward force of the scale F_s. According to Newton's third law F_p and

F_s are equal in magnitude and opposite in direction, so that we need to find F_s in order to find what the scale reads. We can do this, as usual, by applying Newton's second law,

$F_{net} = ma$.

From the free-body diagram we see that $F_{net} = F_s - w$, so that

$F_s - w = ma$.

Solving for F_s gives an equation with only one unknown:

$F_s = ma + w$,

or, because $w = mg$, simply

$F_s = ma + mg$.

No assumptions were made about the acceleration, and so this solution should be valid for a variety of accelerations in addition to the ones in this exercise.

Solution for (a)

In this part of the problem, $a = 1.20\ m/s^2$, so that

$F_s = (75.0\ kg)(1.20\ m/s^2) + (75.0\ kg)(9.80\ m/s2)$,

yielding Fs = 825 N.

Discussion for (a)

This is about 84 kg (=825/9.80). What would the scale have read if he were stationary? Since his acceleration would be zero, the force of the scale would be equal to his weight:

$Fnet = ma = 0 = Fs - w$

$Fs = w = mg$

$Fs = (75.0\ kg)(9.80\ m/s2)$

Fs = 735 N.

So, the scale reading in the elevator is greater than his 735-N (165 lb) weight. This means that the scale is pushing up on the person with a force greater than his weight, as it must in order to accelerate him upward. Clearly, the greater the acceleration of the elevator, greater is the scale reading, consistent with what you feel in rapidly accelerating versus slowly accelerating elevators.

Solution for (b)

Now, what happens when the elevator reaches a constant upward velocity? Will the scale still read more than his weight?

For any constant velocity—up, down, or stationary—acceleration is zero because

$a = \Delta v/\Delta t$, and $\Delta v = 0$.

Thus,

$F_s = ma + mg = 0 + mg.$

Now

F_s = (75.0 kg)(9.80 m/s2), which gives

F_s = 735 N.

Discussion for (b)

The scale reading is 735 N, which equals the person's weight. This will be the case whenever the elevator has a constant velocity—moving up, moving down, or stationary.

The solution to the previous example also applies to an elevator accelerating downward, as mentioned. When an elevator accelerates downward, a is negative, and the scale reading is less than the weight of the person, until a constant downward velocity is reached, at which time the scale reading again becomes equal to the person's weight. If the elevator is in free-fall and accelerating downward at g, then the scale reading will be zero and the person will appear to be weightless.

SELF-TEST 06

(1) A 1000 kg small car with speed 40 km/h collides with a 5000 kg truck moving in opposite direction with a speed 80 km/h . Which of the following statements is true

(A) The car exerts less force on truck in comparison to force exerted by truck on car

(B) The car exerts a force on truck which is equal to force exerted by truck on car

(C) The car exerts more force on truck in comparison to force exerted by truck on car

(D) The data is insufficient to make statement about comparison of forces

(2) A 1000 kg small car with speed 40 km/h collides with a 5000 kg truck moving in opposite direction with a speed 80 km/h . Which of the following statements is true?

(A) the car will suffer heavier damage (compared to truck) due to more force exerted by truck

(B) the car will suffer heavier damage (compared to truck) as speed of truck is more

(C) the car will suffer heavier damage (compared to truck) as weight of truck is more

(D) the car will suffer equal damage (compared to truck) despite the force exerted by it being equal to that exerted by truck.

Short Answer Questions 02

(1) A horse who has read Newton's laws of motion, argues that "no matter how much force I exert on the cart, the cart will exert equal and opposite force on me, hence I will not be able to draw it". How will you convince him to pull the cart?

(2) Explain the scheme of solving the fundamental problem in mechanics as developed by Newton

(3) What is the importance of Newton's laws in discovery of law of gravitation ?

02-03: Dynamics of system of particle

When you have a system of particles you may have two cases. One in which the relative position of the various particles does not change over time during the motion of the system. Second, in which particles do not retain their relative positions with respect to one another. The first one will be called as a rigid body. The examples of rigid body motion could be that of a cricket bat. The various parts of the bat retain their relative positions with respect to one another, even though all of them may have moved over the period of time. The second type of the system may be identified as that of a fluid (liquid or gases)

The motion of a rigid body may be of two kinds: translation and rotation. During the translation motion or linear motion, the all constituent particles of the body move along parallel or straight lines and undergo equal displacements over a given period of time. The velocities of any two constituting particles are same at a given time. In the figure 02-24 it is depicted for the motion of a body. At time $t = 0$, the body is shown as A with various particles labeled as P, Q and R. The body moves to position B, with P, Q, R particles moving to P', Q' and R' respectively over a period of time. Note that PP', QQ' and RR' are parallel to one another. The velocity of P is equal to that of Q and that of R.

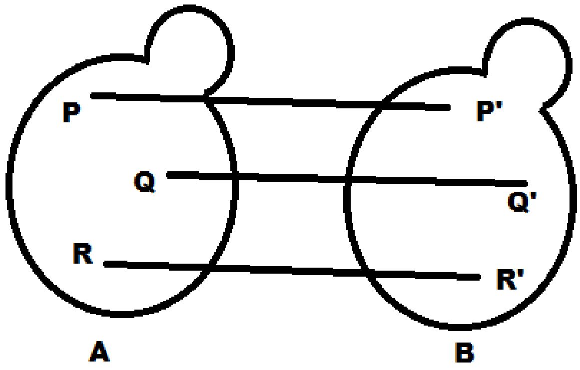


Fig 02.24: Translation motion of a rigid body

The distance PP’=QQ’=RR’ over a infinitesimally small time ∂t can be treated as ∂x.

The velocity of the rigid body can be defined as

$$\boldsymbol{v} = \lim_{t \to 0} \left(\frac{\partial \boldsymbol{x}}{\partial t} \right)$$

(02-01)

The linear acceleration a can be defined as

$$\boldsymbol{a} = \lim_{t \to 0} \left(\frac{\partial \boldsymbol{v}}{\partial t} \right)$$

(02-02)

The rotational motion is characterized by the movement of particles of the rigid body in concentric circles around a line called axis. The linear velocities of the various parts of the body differ significantly. The angular velocities of the various parts of the body however are constant. The rotational motion is produced by a couple or a set of couples equivalent to a single couples.

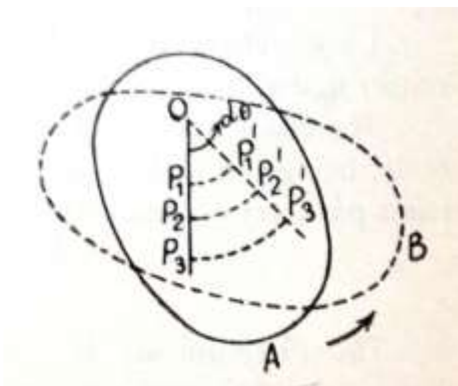


Fig 02.25: Rotational Motion

The figure 02-24 depicts the rotational motion of a rigid body from position A to position B. The axis passes through point O and is perpendicular to the plane of paper.

Particles P1, P2, P3 cover unequal distances P1P'1, P2P'2, P3P'3 in the same interval of time. Thus their linear velocities are different. But during the same duration ∂t, these points rotate by equal angle $\partial\theta$, hence their angular velocity is

$$\boldsymbol{\omega} = \lim_{t \to 0} \left(\frac{\partial \theta}{\partial t}\right) \quad = \frac{d\theta}{dt}\hat{\boldsymbol{\omega}}$$

Here $\hat{\boldsymbol{\omega}}$ is unit vector in the direction of $\boldsymbol{\omega}$.

Angular acceleration is the rate of change of angular velocity: α

$$\alpha = \lim_{t \to 0} \left(\frac{\partial \boldsymbol{\omega}}{\partial t}\right) \quad = \frac{d2\theta}{dt2}\hat{\boldsymbol{\omega}}$$

The analog of mass in rotational system is called moment of inertia. Let us see it in details.

The moment of inertia, otherwise known as the angular mass or rotational inertia, of a rigid body is a physical quantity that determines the torque needed for a desired angular acceleration about a rotational axis; similar to how mass determine the force needed for a desired acceleration. It depends on the body's mass distribution and the axis chosen, with larger moments requiring more torque to change the body's rotation rate.

For a point mass, the moment of inertia is equal to the mass times the square of the distance from the rotation axis. The moment of inertia of a rigid composite system is the sum of the moments of inertia of its component subsystems (all taken about the same axis). For bodies constrained to rotate in a plane, their moment of inertia about an axis perpendicular to the plane is a scalar quantity.

For bodies free to rotate in three dimensions, the situation is extremely complicated and beyond the scope of the present discussion.

Fig 2.26: Flywheel of a steam engine in a factory

When a body is free to rotate around an axis, torque must be applied to change its angular momentum. The amount of torque needed to cause any given angular acceleration (the rate of change in angular velocity) is proportional to the moment of inertia of the body. Moment of inertia may be expressed in units of kilogram meter squared ($kg \cdot m^2$) in SI units.

Moment of inertia plays the role in rotational kinetics that mass (inertia) plays in linear kinetics - both characterize the resistance of a body to changes in its motion. The moment of inertia depends on how mass is distributed around an axis of rotation, and will vary depending on the chosen axis. For a point-like mass, the moment of inertia about some axis is given by mr^2, where r is the distance of the point from the axis, and m is the mass. For an extended rigid body, the moment of inertia is just the sum of all the small pieces of mass multiplied by the square of their distances from the axis in question. For an extended body of a regular shape and uniform density, this summation sometimes produces a simple expression that depends on the dimensions, shape and total mass of the object.

$$I = \sum m_i {r_i}^2$$

Here $m_i$is the mass of ith particle and r_i is the distance of the particle from the axis of rotation.

In 1673, Christiaan Huygens introduced this parameter in his study of the oscillation of a body hanging from a pivot, known as a compound pendulum. The term moment of inertia was introduced by Leonhard Euler in his book *Theoria motus corporum solidorum seu rigidorum* in 1765, and it is incorporated into Euler's second law.

The natural frequency of oscillation of a compound pendulum is obtained from the ratio of the torque imposed by gravity on the mass of the pendulum to the resistance to acceleration defined by the moment of inertia. Comparison of this natural frequency of compound pendulum with that of a simple pendulum, consisting of a single point mass, provide a mathematical formulation for moment of inertia of an extended body.

Moment of inertia I is defined as the ratio of the net angular momentum L of a system to its angular velocity around a principal axis, that is

$$I = \frac{L}{\omega}.$$

(02-03)

If the angular momentum of a system is constant, then as the moment of inertia gets smaller, the angular velocity must increase. This occurs when spinning figure skaters pull in their outstretched arms or divers curl their bodies into a tuck position during a dive, to spin faster.

Fig 2.27: Application of principle of conservation of angular momentum in ice skating

(Source: Wikipedia)

If the shape of the body does not change, then its moment of inertia appears in Newton's law of motion as the ratio of an applied torque **τ** on a body to the angular acceleration **α** around a principal axis, that is

$$\tau = I\alpha.$$

(02-04)

For a simple pendulum, this definition yields a formula for the moment of inertia I in terms of the mass *m* of the pendulum and its distance *r* from the pivot point as,

$$I = mr^2.$$

(02-05)

Thus, moment of inertia depends on both the mass m of a body and its geometry, or shape, as defined by the distance r to the axis of rotation.

This simple formula generalizes to define moment of inertia for an arbitrarily shaped body as the sum of all the elemental point masses *dm*, each multiplied by the square of its perpendicular distance r to an axis k.

In general, given an object of mass m, an effective radius k can be defined for an axis through its center of mass, with such a value that its moment of inertia is

$$I = mk^2,$$

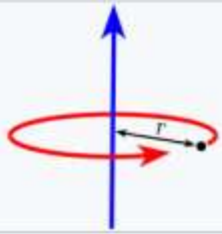
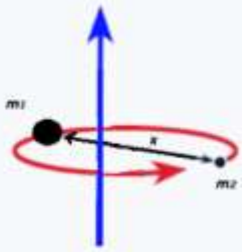
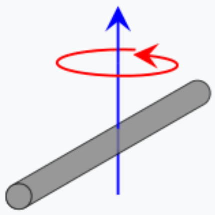
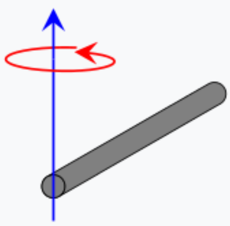
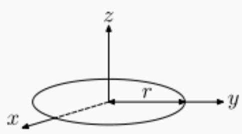
(02-06)

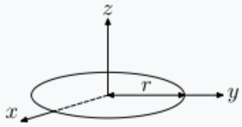
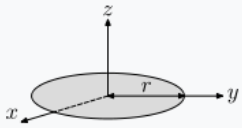
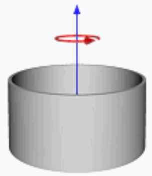
where k is known as the radius of gyration.

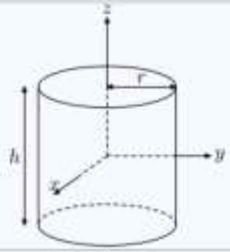
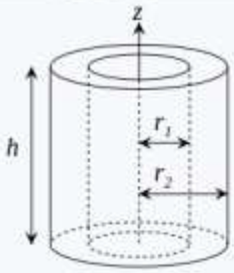
Moment of inertia also appears in momentum, kinetic energy, and in Newton's laws of motion for a rigid body as a physical parameter that combines its shape and mass. There is an interesting difference in the way moment of inertia appears in planar and spatial movement. Planar movement has a single scalar that defines the moment of inertia, while for spatial movement the same calculations yield a 3 × 3 matrix of moments of inertia, called the inertia matrix or inertia tensor.

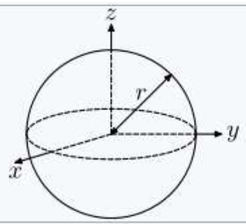
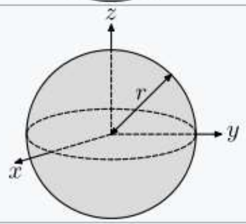
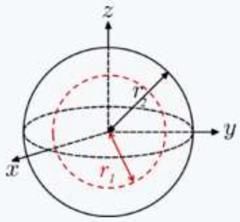
The moment of inertia of a rotating flywheel is used in a machine to resist variations in applied torque to smooth its rotational output. The moment of inertia of an airplane about its longitudinal, horizontal and vertical axis determines how steering forces on the control surfaces of its wings, elevators and tail affect the plane in roll, pitch and yaw.

Moment of Inertia for some cases

Description	Figure	Moment(s) of inertia
Point mass M at a distance r from the axis of rotation. A point mass does not have a moment of inertia around its own axis, but using the parallel axis theorem a moment of inertia around a distant axis of rotation is achieved.		$I = Mr^2$
Two point masses, m_1 and m_2, with reduced mass μ and separated by a distance, x about an axis passing through the center of mass of the system and perpendicular to the line joining the two particles.		$I = \frac{m_1 m_2}{m_1 + m_2} x^2 = \mu x^2$
Rod of length L and mass m, rotating about its center. This expression assumes that the rod is an infinitely thin (but rigid) wire. This is a special case of the thin rectangular plate with axis of rotation at the center of the plate, with $w = L$ and $h = 0$.		$I_{\text{center}} = \frac{1}{12} mL^2$ [1]
Rod of length L and mass m, rotating about one end. This expression assumes that the rod is an infinitely thin (but rigid) wire. This is also a special case of the thin rectangular plate with axis of rotation at the end of the plate, with $h = L$ and $w = 0$.		$I_{\text{end}} = \frac{1}{3} mL^2$ [1]
Thin circular hoop of radius r and mass m. This is a special case of a torus for $a = 0$ (see below), as well as of a thick-walled cylindrical tube with open ends, with $r_1 = r_2$ and $h = 0$.		$I_z = mr^2$ $I_x = I_y = \frac{1}{2} mr^2$

Thin circular hoop of radius r and mass m. This is a special case of a torus for a = 0 (see below), as well as of a thick-walled cylindrical tube with open ends, with $r_1 = r_2$ and h = 0.		$I_z = mr^2$ $I_x = I_y = \frac{1}{2}mr^2$
Thin, solid disk of radius r and mass m. This is a special case of the solid cylinder, with h = 0. That $I_x = I_y = \frac{I_z}{2}$ is a consequence of the perpendicular axis theorem.		$I_z = \frac{1}{2}mr^2$ $I_x = I_y = \frac{1}{4}mr^2$
Thin cylindrical shell with open ends, of radius r and mass m. This expression assumes that the shell thickness is negligible. It is a special case of the thick-walled cylindrical tube for $r_1 = r_2$. Also, a point mass m at the end of a rod of length r has this same moment of inertia and the value r is called the radius of gyration.		$I \approx mr^2$ [1]

Solid cylinder of radius r, height h and mass m. This is a special case of the thick-walled cylindrical tube, with $r_1 = 0$.		$I_z = \frac{1}{2}mr^2$ [1] $I_x = I_y = \frac{1}{12}m\left(3r^2 + h^2\right)$
Thick-walled cylindrical tube with open ends, of inner radius r_1, outer radius r_2, length h and mass m.		$I_z = \frac{1}{2}m\left(r_2^2 + r_1^2\right) = mr_2^2\left(1 - t + \frac{t^2}{2}\right)$ [1] [2] where $t = (r_2 - r_1)/r_2$ is a normalized thickness ratio; $I_x = I_y = \frac{1}{12}m\left(3\left(r_2^2 + r_1^2\right) + h^2\right)$
With a density of ρ and the same geometry		$I_z = \frac{\pi\rho h}{2}\left(r_2^4 - r_1^4\right)$ $I_x = I_y = \frac{\pi\rho h}{12}\left(3(r_2^4 - r_1^4) + h^2(r_2^2 - r_1^2)\right)$

Hollow sphere of radius r and mass m. A hollow sphere can be taken to be made up of two stacks of infinitesimally thin, circular hoops, where the radius differs from 0 to r (or a single stack, where the radius differs from $-r$ to r).		$I = \frac{2}{3}mr^2$ [1]
Solid sphere (ball) of radius r and mass m. A sphere can be taken to be made up of two stacks of infinitesimally thin, solid discs, where the radius differs from 0 to r (or a single stack, where the radius differs from $-r$ to r).		$I = \frac{2}{5}mr^2$ [1]
Sphere (shell) of radius r_2 and mass m, with centered spherical cavity of radius r_1. When the cavity radius r_1 = 0, the object is a solid ball (above). When $r_1 = r_2$, $\left(\frac{r_2^5 - r_1^5}{r_2^3 - r_1^3}\right) = \frac{5}{3}r_2^2$, and the object is a hollow sphere.		$I = \frac{2}{5}m\left(\frac{r_2^5 - r_1^5}{r_2^3 - r_1^3}\right)$ [1]

SOLVED PROBLEMS

Calculate the angular velocity of a 0.300 m radius car tyre when the car travels at 15.0 m/s (about 54 km/h).

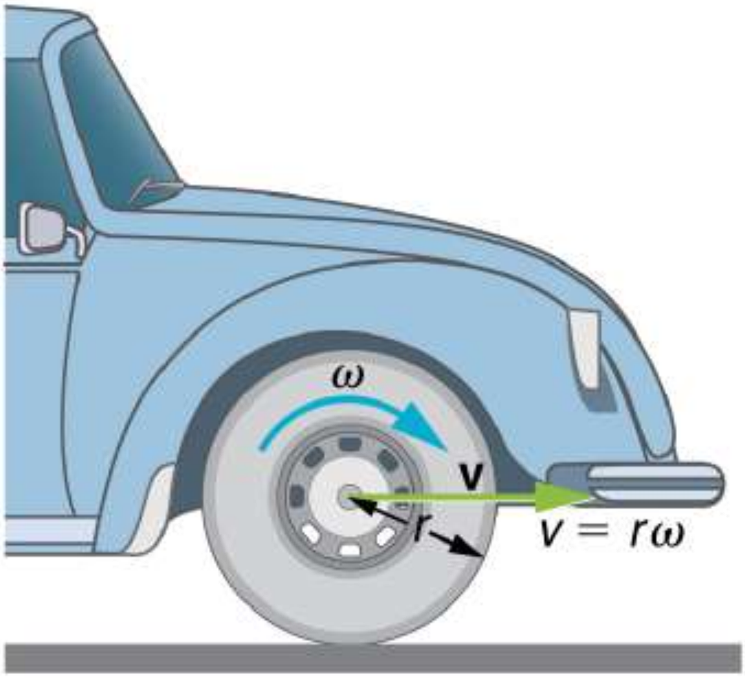


Fig 02-28: A car moving at a velocity v to the right has a tyre rotating with an angular velocity ω .The speed of the tread of the tire relative to the axle is v , the same as if the car were jacked up. Thus the car moves forward at linear velocity v = rω , where r is the tyre radius. A larger angular velocity for the tyre means a greater velocity for the car.

Strategy

Because the linear speed of the tyre rim is the same as the speed of the car, we have v = 15.0 m/s. The radius of the tyre is given to be r = 0.300 m. Knowing v and r, we can use the second relationship in v = rω, ω = vr to calculate the angular velocity.

Solution

To calculate the angular velocity, we will use the following relationship:

ω = vr.

Substituting the known quantities,

ω = 15.0 m/s

0.300 m = 50.0 rad/s.

Discussion

When we cancel units in the above calculation, we get 50.0/s. But the angular velocity must have units of rad/s. Because radians are actually unit less (radians are defined as a ratio of distance), we can simply insert them into the answer for the angular velocity. Also note that if an earth mover with much larger tyres, say 1.20 m in radius, were moving at the same speed of 15.0 m/s, its tyres would rotate more slowly. They would have an angular velocity

$\omega = (15.0 \text{ m/s}) / (1.20 \text{ m}) = 12.5 \text{ rad/s}$.

Both ω and v have directions (hence they are angular and linear velocities, respectively). Angular velocity has only two directions with respect to the axis of rotation—it is either clockwise or counterclockwise. Linear velocity is tangent to the path, as illustrated in Figure 02-08.

Self-Test 07

(1) A body moves in such a way that there are some points on the body whose coordinates do not change. Which of the following is true?

(A) the body is undergoing rotational motion

(B) the body is undergoing translational motion

(C) there is a contradiction in statement

(D) this is an example of inertial frame of reference.

(2) A body moves in such a way that there are no points on the body whose coordinates do not change. Which of the following is true?

(A) the body is undergoing rotational motion

(B) the body is undergoing translational motion

(C) there is a contradiction in statement

(D) this is an example of non-inertial frame of reference.

Short Answer Questions 03

(1) Explain why rotating systems are always accelerated ones.

(2) Explain how the inner tyres of a car which is making a turn have a linear speed different than that of the outer tyres. What will be effect of such difference of speeds and how is it taken care in a car?

02-04: Centre of Mass

In physics, the center of mass of a distribution of mass in space is the unique point where the weighted relative position of the distributed mass sums to zero, or the point where if a force is applied it moves in the direction of the force without rotating. The distribution of mass is balanced around the center of mass and the average of the weighted position coordinates of the distributed mass defines its coordinates. Calculations in mechanics are often simplified when formulated with respect to the center of mass. It is a hypothetical point where entire mass of an object may be assumed to be concentrated to visualize its motion. In other words, the center of mass is the particle equivalent of a given object for application of Newton's laws of motion.

In the case of a single rigid body, the center of mass is fixed in relation to the body, and if the body has uniform density, it will be located at the centroid. The center of mass may be located outside the physical body, as is sometimes the case for hollow or open-shaped objects, such as a horseshoe. In the case of a distribution of separate bodies, such as the planets of the Solar System, the center of mass may not correspond to the position of any individual member of the system.

The center of mass is a useful reference point for calculations in mechanics that involve masses distributed in space, such as the linear and angular momentum of planetary bodies and rigid body dynamics. In orbital mechanics, the equations of motion of planets are formulated as point masses located at the centers of mass. The center of mass frame is an inertial frame in which the center of mass of a system is at rest with respect to the origin of the coordinate system.

Fig 2.29: This toy applies the principle of center of mass

(Source: Wikipedia)

HISTORICAL BACKGROUND

The concept of "center of mass" in the form of the center of gravity was first introduced by the ancient Greek physicist, mathematician, and engineer Archimedes of Syracuse. He worked with simplified assumptions about gravity that amount to a uniform field, thus arriving at the mathematical properties of what we now call the center of mass. Archimedes showed that the torque exerted on a lever by weights resting at various points along the lever is the same as what it would be if all of the weights were moved to a single point—their center of mass. In work on floating bodies he demonstrated that the orientation of a floating object is the one that makes its center of mass as low as possible. He developed mathematical techniques for finding the centers of mass of objects of uniform density of various well-defined shapes.

Later mathematicians who developed the theory of the center of mass include Pappus of Alexandria, Guido Ubaldi, Francesco Maurolico, Federico Commandino, Simon Stevin, Luca Valerio, Jean-Charles de la Faille, Paul Guldin, John Wallis, Louis Carré, Pierre Varignon, and Alexis Clairaut.

Newton's second law is reformulated with respect to the center of mass in Euler's first law.

DEFINITION

The center of mass is the unique point at the center of a distribution of mass in space that has the property that the weighted position vectors relative to this point sum to zero. In analogy to statistics, the center of mass is the mean location of a distribution of mass in space.

A SYSTEM OF PARTICLES

In the case of a system of particles P_i, i = 1, …, n , each with mass m_i that are located in space with coordinates r_i, i = 1, …, n , the coordinates R of the center of mass satisfy the condition

$$\sum_{i=1}^{n} m_i(\mathbf{r}_i - \mathbf{R}) = 0.$$

(02-07)

Solving this equation for R yields the formula

$$\mathbf{R} = \frac{1}{M}\sum_{i=1}^{n} m_i \mathbf{r}_i,$$

(02-08)

where M is the sum of the masses of all of the particles.

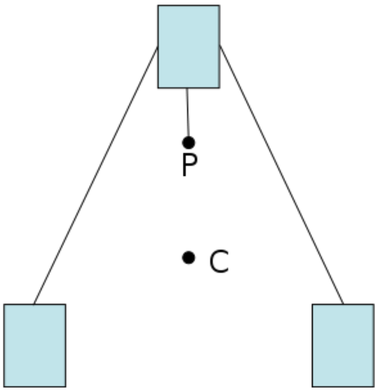


Fig 02.30: Diagram of an educational toy that balances on a point: the center of mass (C) settles below its support (P)

A CONTINUOUS VOLUME

If the mass distribution is continuous with the density ρ(r) within a solid Q, then the integral of the weighted position coordinates of the points in this volume relative to the center of mass R over the volume V is zero, that is

$$\iiint_Q \rho(\mathbf{r})(\mathbf{r} - \mathbf{R})dV = 0.$$

(02-09)

Solve this equation for the coordinates R to obtain

$$\mathbf{R} = \frac{1}{M}\iiint_Q \rho(\mathbf{r})\mathbf{r}dV,$$

(02-10)

where M is the total mass in the volume.

If a continuous mass distribution has uniform density, which means ρ is constant, then the center of mass is the same as the centroid of the volume.

LOCATING THE CENTER OF MASS

The experimental determination of the center of mass of a body uses gravity forces on the body and relies on the fact that in the parallel gravity field near the surface of the earth the center of mass is the same as the center of gravity.

The center of mass of a body with an axis of symmetry and constant density must lie on this axis. Thus, the center of mass of a circular cylinder of constant density has its center of mass on the axis of the cylinder. In the same way, the center of mass of a spherically symmetric body of constant density is at the center of the sphere. In general, for any symmetry of a body, its center of mass will be a fixed point of that symmetry.

IN TWO DIMENSIONS

An experimental method for locating the center of mass is to suspend the object from two locations and to drop plumb lines from the suspension points. The intersection of the two lines is the center of mass.

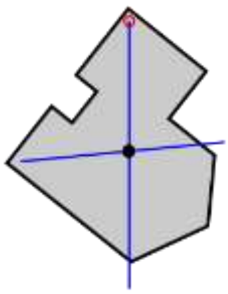


Fig 02.31: Plumb line method.

The shape of an object might already be mathematically determined, but it may be too complex to use a known formula. In this case, one can subdivide the complex shape into simpler, more elementary shapes, whose centers of mass are easy to find. If the total mass and center of mass can be determined for each area, then the center of mass of the whole is the weighted average of the centers. This method can even work for objects with holes, which can be accounted for as negative masses.

A direct development of the planimeter known as an integraph, or integerometer, can be used to establish the position of the centroid or center of mass of an irregular two-dimensional shape. This method can be applied to a shape with an irregular, smooth or complex boundary where other methods are too difficult. It was regularly used by ship builders to compare with the required displacement and center of buoyancy of a ship, and ensure it would not capsize.

Solved Problems

Example (1) Centre of mass for a linear strip using home experiment

Find the centre of mass of a linear strip with negligible width

Solution

By intuition (without using mathematics) we can find the CM for a strip or a wire or a pencil by following the simple experiment.

Hold the pen or pencil in your hand at its one end (say its writing tip)and release your grip. It will fall on one side as the fig 02-32shows. It means that the CM (where all the mass of pen is assumed to be concentrated) should be somewhere on other side as the gravity is pulling it.

I may repeat this experiment on other side of the pen. It will droop down towards the writing tip side. Again I would argue that CG lies on a point which is between my grip and writing tip.

If I hold the pen at the midpoint and loosen my grip, I find that it does not drop on either side. Hence the mass of the pen may be thought to be concentrated on the midpoint of the pen.

Since I am assuming the pen to be of negligible diameter, only parameter of the pen to be specified for locating the CM is the distance from one end (say writing tip) to the CM. If the pen has a length L, the CM lies at L/2 from tip.

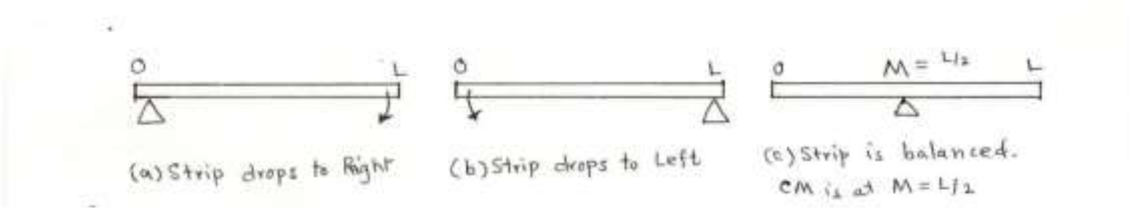


Fig 02-32: Centre of mass of a strip

Example (2) CM of a Rectangle (without maths)

I am given a rectangle of length L and Width W, and I have to find the coordinates of CM.

Solution

I have found in the previous example that the CM for a strip of length L (and zero width) lies on the midpoint of the strip which lies at L/2 distance from an end. Now my problem is that I have a figure with length and (non-zero) width.

To make use of the foregoing conclusion, I divide the rectangle into a number N of strips by drawing lines parallel to its lengths. I will have N strips with length L and width L/N. If N is large number width of each of the strips would be negligible.

The CM of each of the strip should lie on the midpoints M for each of the strips. If I assemble such strips on the rectangle, I would conclude that the CM should lie on a line which passes through midpoints of all such strips, i.e., a line which is parallel to width and is at a distance L/2 from either of the ends from which length is measured.

Such line is again a strip with zero width and length W. From previous example, I conclude that the CM for this strip should lie on its mid point (W/2)

Hence the CM for the rectangular lamina should be from L/2 from the length ends and W/2 from width ends, thus the coordinates would be (L/2, W/2)

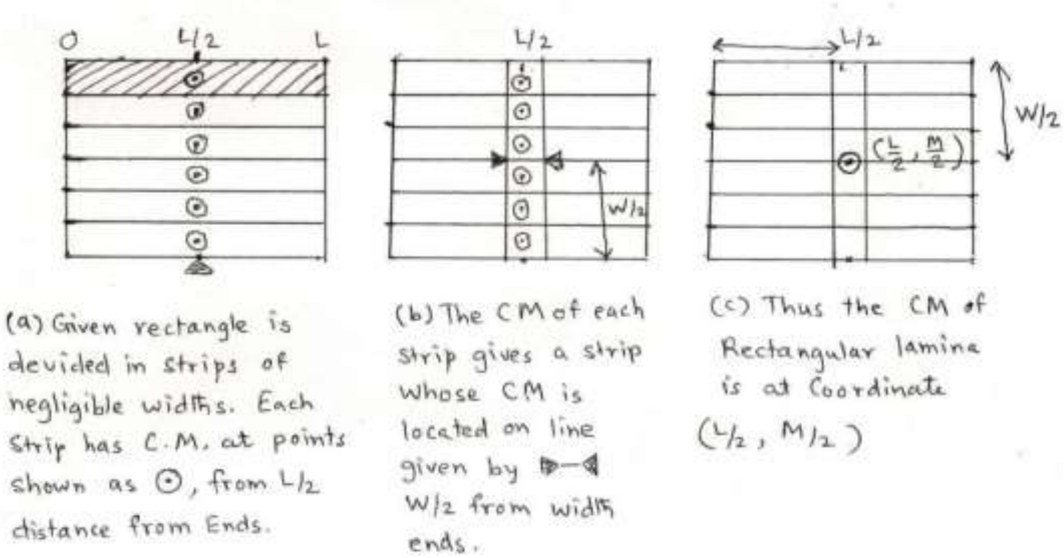


Fig 2-33: Finding CM of a rectangular lamina

SELF-TEST 08

(1) The point on the body where all the mass can be thought to be concentrated is called

(A) Centre of Mass

(B) Centre of gyration

(C) Centre of symmetry

(D) Axis of rotation

(2) Which of the following can be determined using a plumb line?

(A) Centre of Mass

(B) Centre of gyration

(C) Centre of buoyancy

(D) Axis of rotation

Short Answer Questions 04

(1) Explain how the concept of centre of mass is used in design of a physical balance.

(2) What is the importance of moment of inertia?

(3) State the Newton's laws of motion for rotational motion.

Key Words

Frames of reference, Inertial and non-inertial frames, Newton's Laws of Motion, Momentum, Force, Action-reaction dualism, Conservation Laws, translation motion, rotation motion, Momentum of inertia, angular momentum, angular displacement, angular velocity, torque, radius of gyration, centre of mass.

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Course Companion Website

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THEORY CREDIT 02

UNIT 02-01: VECTOR ALGEBRA

LEARNING OBJECTIVES

After successful completion of this unit, you will be able to

- Explain the concept like vector, scalar and derivatives of vectors
- Calculate the vector triple products
- Calculate the derivatives of a vector with respect to a scalar parameter

INTRODUCTION

Fig 01.01: The motion of a bird like this American kestrel is described by vector quantities like displacement, velocity and acceleration

What is the difference between distance and displacement? Whereas displacement is defined by both direction and magnitude, distance is defined only by magnitude. Displacement is an example of a vector quantity. Distance is an example of a scalar quantity. A vector is any quantity with both magnitude and direction. Other examples of vectors include a velocity of 90 km/h east and a force of 500 newton straight down.

The direction of a vector in one-dimensional motion is given simply by a plus (+) or minus (−) sign. Vectors are represented graphically by arrows. An arrow used to represent a vector has a length proportional to the vector's magnitude (e.g., the larger the magnitude, the longer the length of the vector) and points in the same direction as the vector.

Some physical quantities, like distance, either have no direction or none is specified. A scalar is any quantity that has a magnitude, but no direction. For example, a 20ºC temperature, 250 Calories of energy in a candy bar, a 90 km/h speed limit, a person's 1.8 m height, and a distance of 2.0 m are all scalars - quantities with no specified direction. Note, however, that a scalar can be negative, such as a −20ºC temperature. In this case, the minus sign indicates a point on a scale rather than a direction. Scalars are never represented by arrows.

In the present unit I will take you through the fine points of the vector algebra.

01-01: Vectors and Vector Algebra

Scalars

A physical quantity which is completely described by a single real number is called a scalar. Physically, it is something which has a magnitude, and is completely described by this magnitude. Examples are temperature, density and mass. In the following, lowercase (usually Greek) letters, e.g. γ,β, α, will be used to represent scalars.

Vectors

The concept of the vector is used to describe physical quantities which have both, a magnitude and a direction associated with them. Examples are force, velocity, displacement and acceleration.

However, this definition is very basic and has many flaws. More rigorous definition is through the various defining properties as given under the sub-section of Vector Algebra below.

Geometrically, a vector is represented by an arrow; the arrow defines the direction of the vector and the magnitude of the vector is represented by the length of the arrow, as shown in Fig 01-02. Analytically, vectors will be represented by lowercase bold-face Latin letters, e.g. **a, r, q**.

The ***magnitude*** (or ***length***) of a vector is denoted by a. It is a scalar and must be non-negative. Any vector whose length is 1 is called a unit vector; unit vectors will usually be denoted by **e**.

(a) (b)

Fig 01-02: (a) Vector (b) Addition of vectors

Vector Algebra

The operations of addition, subtraction and multiplication familiar in the algebra of numbers (or scalars) can be extended to algebra of vectors.

The following definitions and properties fundamentally define the vector:

1. Sum of Vectors:

The addition of vectors **a** and **b** is a vector **c** formed by placing the initial point of b on the terminal point of a and then joining the initial point of a to the terminal point of b. The sum is written as **c =a+b**. This definition is called the parallelogram law for vector addition because, in a geometrical interpretation of vector addition, c is the diagonal of a parallelogram formed by the two vectors **a** and **b**, in Fig. 01-02 b. The following properties hold for vector addition:

a+b =b+a… commutative law

a+(b+c)=(a+b)+c… associative law

2. The Negative Vector:

For each vector **a** there exists a negative vector. This vector has direction opposite to that of vector **a** but has the same magnitude; it is denoted by **-a** . A geometrical interpretation of the negative vector is shown in Fig.01.03a.

3. Subtraction of Vectors and the Zero Vector:

The subtraction of two vectors **a** and **b** is defined by **a - b = a + (-b)** , Fig01.03(b).

If **a = b** then **a - b** is defined as the zero vector (or null vector) and is represented by the symbol **o**. It has zero magnitude and unspecified direction. A proper vector is any vector other than the null vector. Thus the following properties hold:

a+0 = **a**

a + (- **a**) = 0

4. SCALAR MULTIPLICATION:

The product of a vector **a** by a scalar α is a vector α**a** with magnitude α times the magnitude of **a** and with direction the same as or opposite to that of **a**, according as α is positive or negative. If α = 0 , α a is the null vector. The following properties hold for scalar multiplication:

(α+β)**a** = α**a** + β **b** … ..distributive law, over addition of scalars

α (**a** + **b**) = α**a** + β**b**…distributive law, over addition of vectors

α (β**a**) = (αβ)**a**………associative law for scalar multiplication

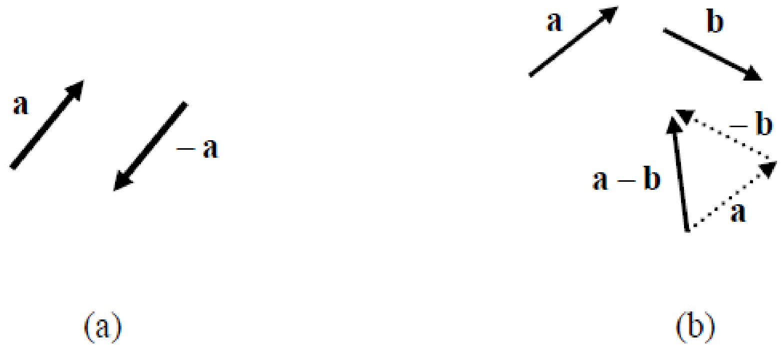

Fig 01-03 (a) negative of a vector; (b) subtraction of vectors

Note that when two vectors **a** and **b** are equal, they have the same direction and magnitude, regardless of the position of their initial points. Thus a = b in Fig. 01-04. A particular position in space is not assigned here to a vector – it just has a magnitude and a direction. Such vectors are called free, to distinguish them from certain special vectors to which a particular position in space is actually assigned.

Fig. 01.04: equal vectors

The vector as something with "magnitude and direction" and defined by the above rules is an element of one case of the mathematical structure, the vector space.

THE DOT PRODUCT

The dot product of two vectors **a** and **b** (also called the scalar product) is denoted by

a. b .

It is a scalar defined by

a.**b** = |**a**||**b**| cosθ .

(01.01)

Here θ is the angle between the vectors when their initial points coincide and is restricted to the range $0 \leq \theta \leq \pi$, Fig. 01.05.

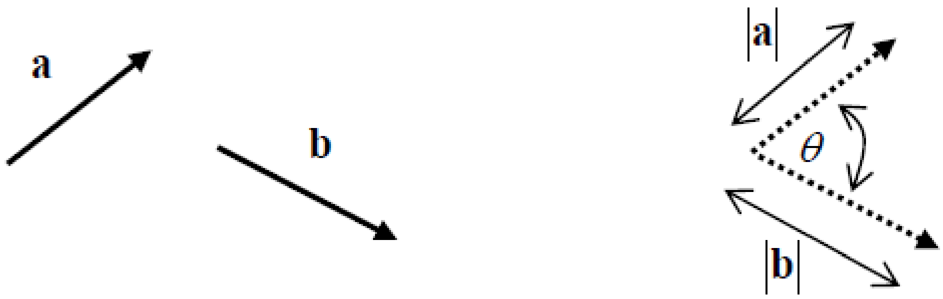

Fig. 01.05: the dot product

An important property of the dot product is that if for two (proper) vectors **a** and **b**, the relation **a**.**b** = **0** , then **a** and **b** are perpendicular. The two vectors are said to be orthogonal. Also, **a**.**a** = |**a**| |**a**| cos(0) , so that the length of a vector is |**a**| = $\sqrt{\boldsymbol{a}\,.\,\boldsymbol{a}}$.

Another important property is that the projection of a vector **u** along the direction of a unit vector **e** is given by **u . e** . This can be interpreted geometrically as in Fig. 01-06.

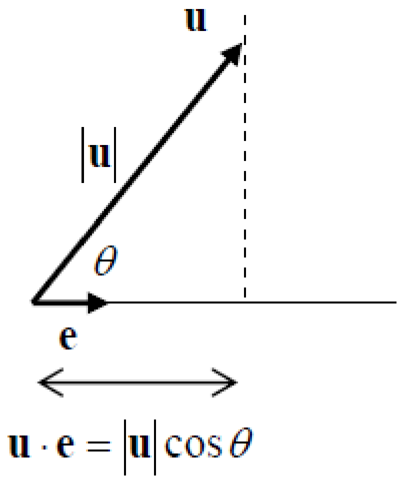

Figure 01-06: Projection of a vector along the direction of a unit vector

It follows that any vector **u** can be decomposed into a component parallel to a (unit) vector **e** and another component perpendicular to **e**, according to

$$\mathbf{u} = (\mathbf{u} \cdot \mathbf{e})\mathbf{e} + [\mathbf{u} - (\mathbf{u} \cdot \mathbf{e})\mathbf{e}]$$

(01-02)

The dot product possesses the following properties (which can be proved using the above definition):

$$
\begin{array}{lll}
(1) & \mathbf{a}\cdot\mathbf{b}=\mathbf{b}\cdot\mathbf{a} & \text{(commutative)}\\
(2) & \mathbf{a}\cdot(\mathbf{b}+\mathbf{c})=\mathbf{a}\cdot\mathbf{b}+\mathbf{a}\cdot\mathbf{c} & \text{(distributive)}\\
(3) & \alpha(\mathbf{a}\cdot\mathbf{b})=\mathbf{a}\cdot(\alpha\mathbf{b}) & \\
(4) & \mathbf{a}\cdot\mathbf{a}\geq 0;\ \text{and}\ \mathbf{a}\cdot\mathbf{a}=0\ \text{if and only if}\ \mathbf{a}=\mathbf{o} &
\end{array}
$$

THE CROSS PRODUCT

The cross product of two vectors **a** and **b** (also called the vector product) is denoted by **a**x**b** . It is a vector with magnitude

$$|\mathbf{a}\times\mathbf{b}|=|\mathbf{a}||\mathbf{b}|\sin\theta$$

(01-03)

with θ defined as for the dot product. It can be seen from the Figure 01-07 that the magnitude

of **a** × **b** is equivalent to the area of the parallelogram determined by the two vectors **a** and **b**.

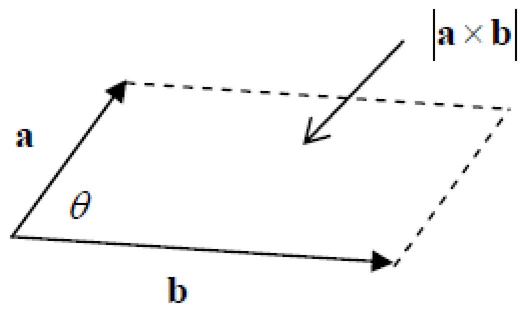

Figure 01-07: the magnitude of the cross product

The direction of this new vector is perpendicular to both **a** and **b**. Whether **a** × **b** points

"up" or "down" is determined, from the fact that the three vectors a, b and a × b form a right handed system. This means that if the thumb of the right hand is pointed in the direction of **a** × **b** , and the open hand is directed in the direction of **a**, then the curling of the fingers of the right hand so that it closes should move the fingers through the angle θ, $0 \leq \theta \leq \pi$, bringing them to **b**.

Some examples are shown in Fig. 01-08.

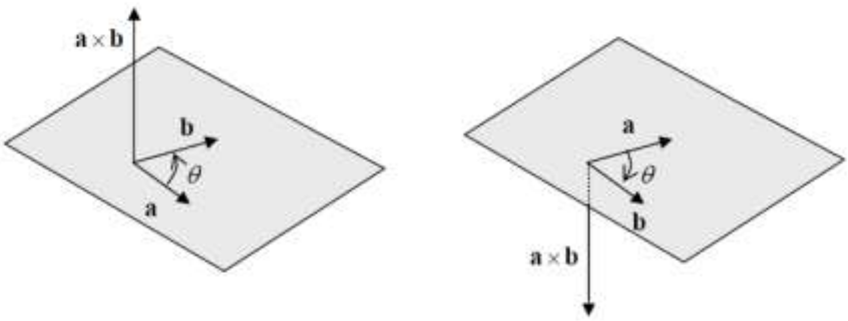

Figure 01-08: Examples of the cross product

The cross product possesses the following properties (which can be proved using the above definition):

(1) $\mathbf{a}\times\mathbf{b} = -\mathbf{b}\times\mathbf{a}$ (*not* commutative)
(2) $\mathbf{a}\times(\mathbf{b}+\mathbf{c}) = \mathbf{a}\times\mathbf{b}+\mathbf{a}\times\mathbf{c}$ (distributive)
(3) $\alpha(\mathbf{a}\times\mathbf{b}) = \mathbf{a}\times(\alpha\mathbf{b})$
(4) $\mathbf{a}\times\mathbf{b} = \mathbf{o}$ if and only if $\mathbf{a}$ and $\mathbf{b}$ $(\neq \mathbf{o})$ are parallel (“linearly dependent”)

The Triple Scalar Product

The triple scalar product, or box product, of three vectors **u, v, w** is defined by

$$\boxed{(\mathbf{u}\times\mathbf{v})\cdot\mathbf{w} = (\mathbf{v}\times\mathbf{w})\cdot\mathbf{u} = (\mathbf{w}\times\mathbf{u})\cdot\mathbf{v}}$$

(01-04)

Let me point out that the order of the vectors **u,v,w** is important. In the above equation the sequence of the vectors is **u-v-w**, **v-w-u**, **w-u-v**. This sequence is *cyclic*. If you draw a triangle with vertices **u,v,w** then in moving from **u** to **v** to **w** you are moving in the same ‘direction’ (clockwise/anticlockwise) as you would be moving from v to w to u or that in case of **w** to **u** to **v**. On the other hand **(u×v).w** is not equal to **(w×v).u**.

Here I introduce the idea of right-handed triad. A Three ordered non-coplanar vectors u,v,w which have a common initial point, are said to form a right-handed or dextral system (Latin dexter = ‘right’), if a right-threaded screw rotated through an angle less than 180° from **u** to **v** will advance in the direction of **w**. For instance, the usual basis vectors i,j and k form a right-handed (dextral) system.

Its importance lies in the fact that, if the three vectors form a right-handed triad, then the volume V of a parallelepiped spanned by the three vectors is equal to the box product.

To see this, let **e** be a unit vector in the direction of **u** × **v** , Fig. 01-09. Then the projection of **w** on **u** × **v** is h = **w** × **e** , and

$$\mathbf{w}\cdot(\mathbf{u}\times\mathbf{v}) = \mathbf{w}\cdot(|\mathbf{u}\times\mathbf{v}|\mathbf{e})$$
$$= |\mathbf{u}\times\mathbf{v}|h$$
$$= V$$

(02-01-05)

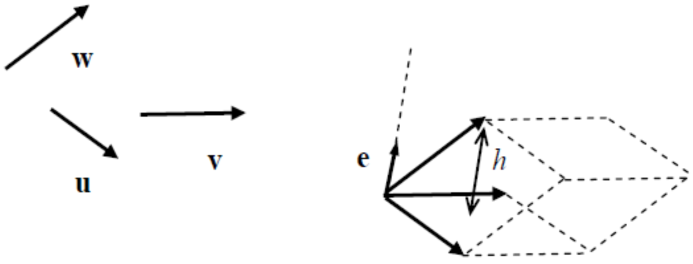

Fig. 01-09: the triple scalar product

Note: if the three vectors do not form a right handed triad, then the triple scalar product yields the negative of the volume. For example, using the vectors above, **(w×v).u =-**V.

Vectors and Points

Vectors are objects which have magnitude and direction, but they do not have any specific location in space. On the other hand, a point has a certain position in space, and the only characteristic that distinguishes one point from another is its position. Points cannot be "added" together like vectors. On the other hand, a vector **v** can be added to a point **p** to give a new point **q**, **q** = **v**+**p**, Fig. 1.10. Similarly, the "difference" between two points gives a vector, **q** = **p**-**v**. Note that the notion of point as defined here is slightly different to the familiar point in space with axes and origin – the concept of origin is not necessary for these points and their simple operations with vectors.

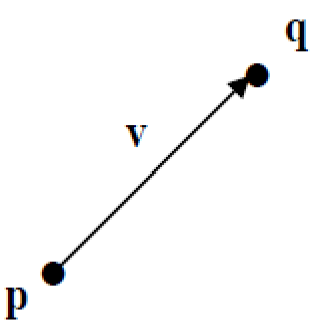

Figure 01-10: adding vectors to points

SOLVED PROBLEMS

Find the magnitude of the sum of three unit vectors drawn from a common vertex of a cube along three of its sides.

Solution: Let i,j,k be the three unit vectors in the direction of x-, y- and z- axis. The required vector is r=i+j+k. Using Pythagoras Theorem

$|r|^2 = (|i|+|j|)^2+|k|^2 = (|i|^2+|j|^2)+|k|^2 = 1^2+1^2+1^2 = 3$

Hence $|r| = \sqrt{3}$

Consider two non-collinear (not parallel) vectors **a** and **b**. Show that a vector **r** lying in the same plane as these vectors can be written in the form **r** = p**a** + q**b**, where p and q are scalars. [Note: one says that all the vectors r in the plane are specified by the base vectors **a** and **b**.]

Solution:

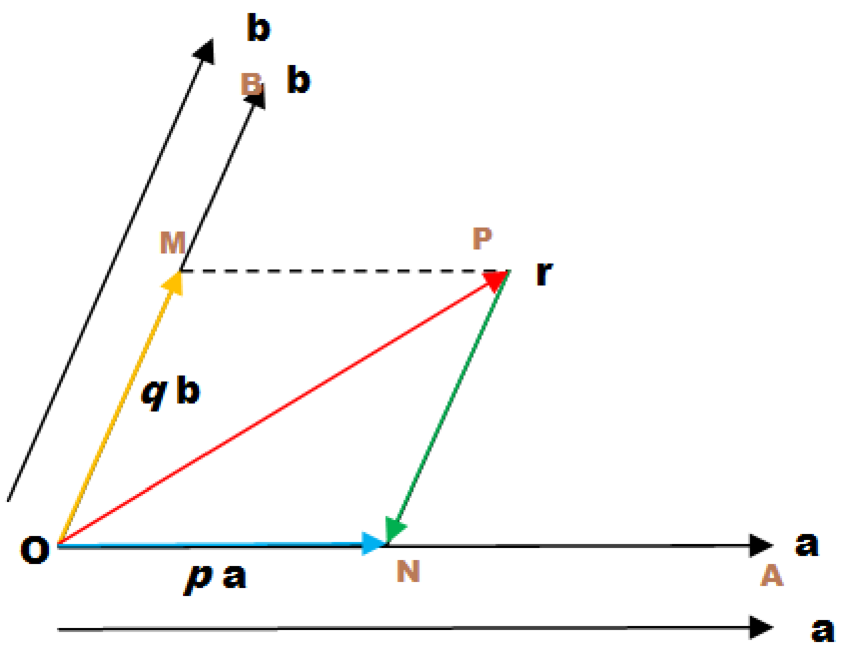

Fig 01-11: Construction to show that **r** = p**a**+q**b** can be always constructed for non-collinear vectors **a** and **b** in a plane containing **r**.

You have been given two non-collinear vectors **a** and **b**. Draw them in the plane.

Let **r** be an arbitrary vector coplanar (in the same plane as) **a** and **b**. Draw this arbitrary vector **r** in the plane as OP.

Draw the vectors OA and OB (equal to **a** and **b**) on the plane such that the initial points of **a** and **b** coincide with the initial point of **r**.

You may now draw PN parallel to OB and PM parallel to OA, so that vector r (OP) is equal to addition of vectors ON and OM.

Since ON is collinear with OA and OM is collinear with OB, there exist scalars p and q such that ON =p**a** and OM =q**b.**

As vector **r** (OP) is equal to addition of vectors ON and OM.

Hence r = p**a** +q**b**.

SELF-TEST 09

1) Which of the following are scalars?

(i) Weight

(ii) Specific heat

(iii) Momentum

(iv) Energy

(v) Volume

Choose the options:

Option A: (ii),(iv),(v)

Option B (i),(iii)

Option C (i),(ii),(iii)

Option D (iii),(iv),(v)

(2)Which of the vector does not have specific direction?

(A)Unit vector

(B) Zero vector

(C)Proper vector

(D) None of the above (all vectors have specific directions)

SHORT ANSWER QUESTIONS 01

(1) What is the direction of a zero vector? If a zero vector does not have any (specific) direction, can it be called a vector?

(2) Can two vectors of different magnitude be added to get a zero resultant? Can three vectors of different magnitude be added to get a zero resultant?

(3) Do the commutative and associative laws apply to vector subtractions?

(4)Can a scalar product be negative quantity?

01-02: Derivative of a Vector with respect to a Parameter

A vector in three dimensions can be resolved in terms of three unit vectors i, j and k collinear with the x-, y- and z- azis. Thus an arbitrary vector **r** may be written as

r = m**i**+n**j**+p**k**

The scalars m, n and p uniquely represents the vector **r**.

A physical quantity may have variations with respect to any parameter. That is the quantity Q may be a function of another quantity s. You may recall that we call a correspondence as 'function' if it is single-valued (that is for every value of independent variable t there is only one value of f(t))

We can study these concepts of calculus for the physical quantities which are vector also.

One of the most basic vector quantities is the position vector. Position of a particle is specified in 3 D as three numbers (x,y,z).

If we see the variation of position vector with respect to time, we can get the derivative of the position vector by differentiating each of the components with respect to time:

$$v_x = \frac{\partial x}{\partial t}, v_y = \frac{\partial y}{\partial t}, v_z = \frac{\partial z}{\partial t}$$

(02-01-06)

The triple (v_x, v_y, v_z) is called as velocity vector.

For example, let's start with a relatively simple vector-valued position vector function **s**(t), with only two components,

$$\vec{\mathrm{s}}(t) = \begin{bmatrix} 2\sin(t) \\ 2\cos(t/3)t \end{bmatrix}$$

We have represented the vector as a column matrix. The element at first row and first column is the x component of position and other element is the y- component. Thus x=2sin(t) and y=2t cost(t/3).

This is parametric representation of the function. If we plot it on paper it will look like the following:

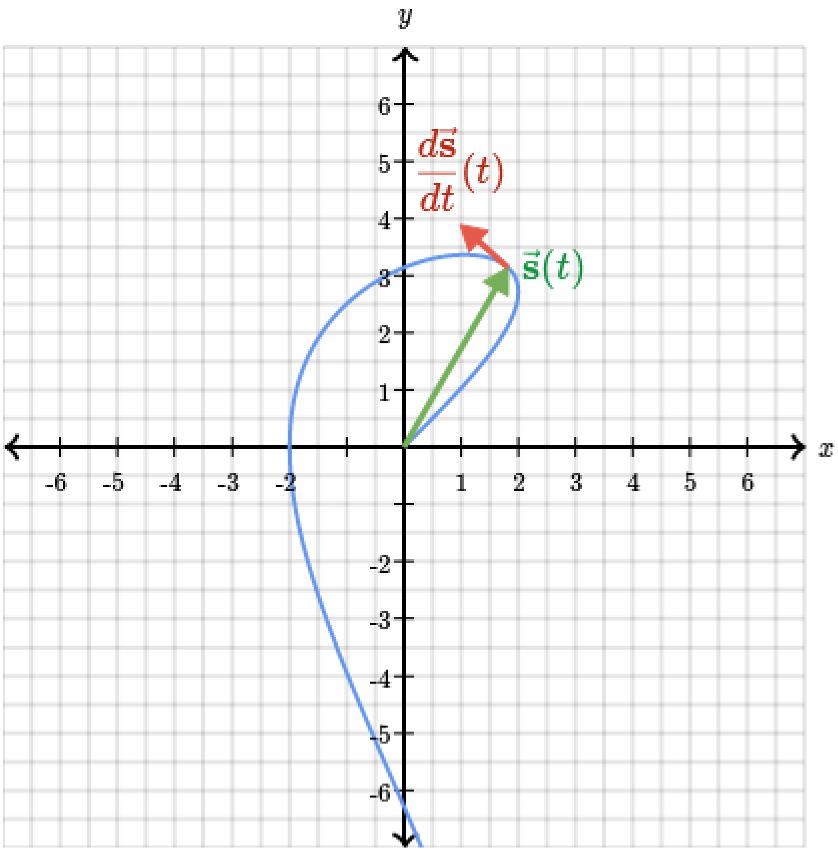

Fig 01-12: Position vector for the example

We can find the velocity vector by taking derivative of each of the component

$$\frac{d\vec{s}}{dt}(t) = \begin{bmatrix} \frac{d}{dt}(2\sin(t)) \\ \frac{d}{dt}(2\cos(t/3))t \end{bmatrix}$$

$$= \begin{bmatrix} 2\cos(t) \\ 2\cos(t/3) - \frac{2}{3}\sin(t/3)t \end{bmatrix}$$

We can write the derivative as s'(t) and identify it as velocity.

$$\vec{s}'(t) = \begin{bmatrix} x'(t) \\ y'(t) \end{bmatrix}$$

If we take t =2, the value of the position at time 2 sec can be found:

$$\vec{s}(2) = \begin{bmatrix} 2\sin(2) \\ 2\cos(2/3)\cdot 2 \end{bmatrix} \approx \begin{bmatrix} 1.819 \\ 3.144 \end{bmatrix}$$

It can be seen on a graph paper as follows

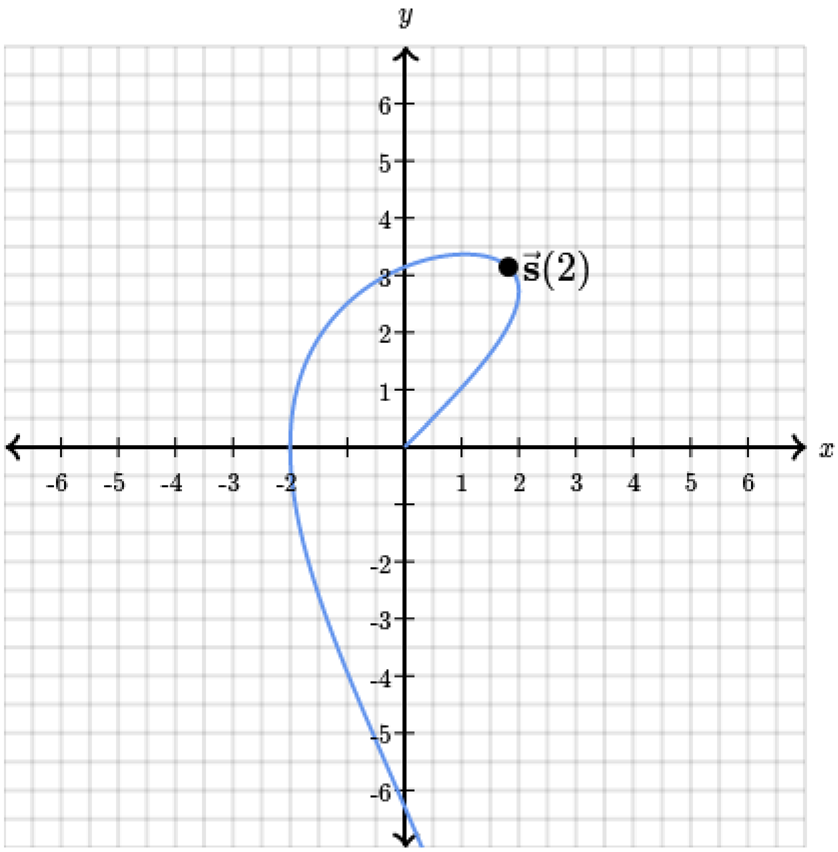

Fig 01-13: The position of the particle at t=2

The velocity at time 2 can also be found as

$$\frac{d\vec{s}}{dt}(2) = \begin{bmatrix} 2\cos(2) \\ 2\cos(2/3) - \frac{2}{3}\sin(2/3)\cdot 2 \end{bmatrix} \approx \begin{bmatrix} -0.832 \\ 0.747 \end{bmatrix}$$

If we plot it on the paper it will look like this.

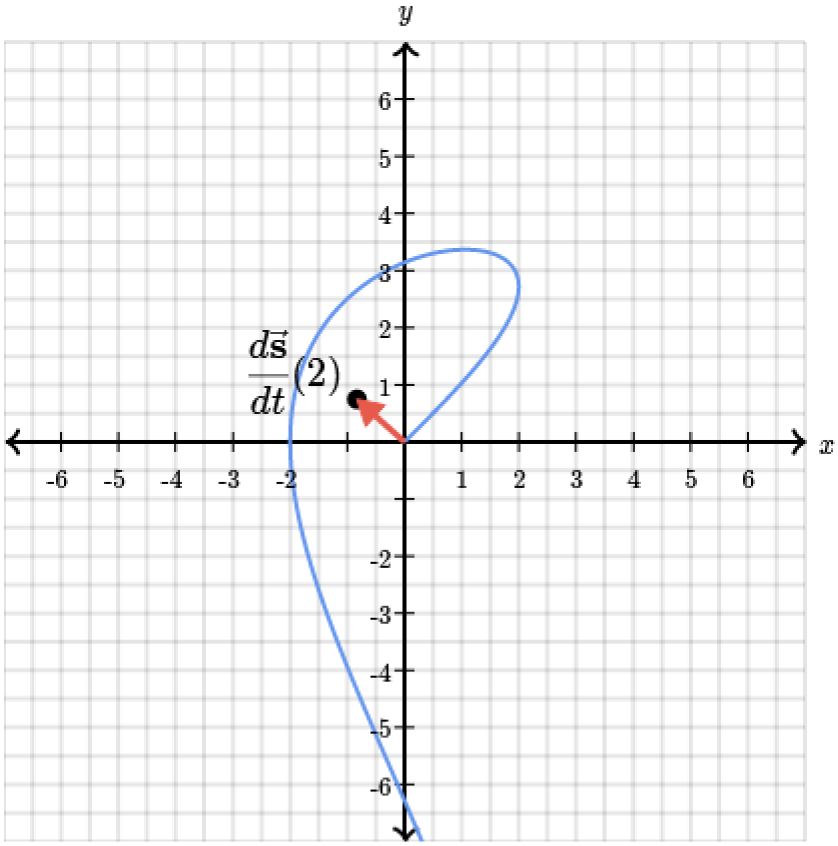

Fig 01-14: Plotting velocity vector at origin

It's hard to see what this derivative vector represents when it just sits at the origin, but if we shift it so that it's tail sits on the tip of the vector $\vec{s}$, it has a wonderful interpretation:

If $\mathbf{s}(t)$ represents the position of a traveling particle as a function of time, $\frac{ds}{dt}$ is the velocity vector of that particle at time t_0.

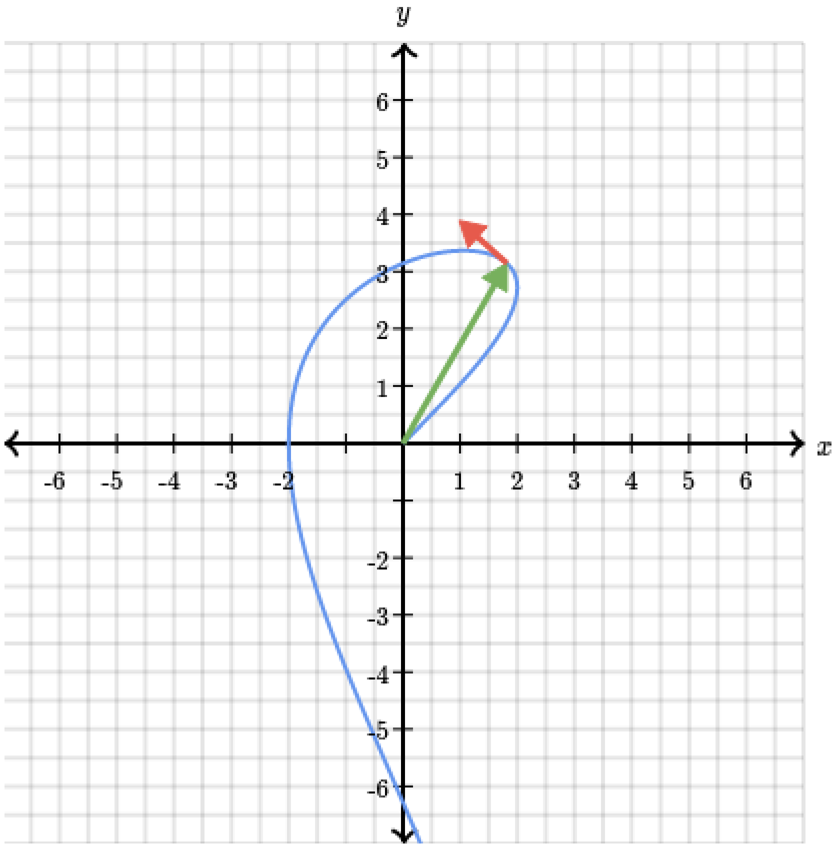

Fig 01-15: Tip of Position vector as function of time with value of s(2) and s'(2) shown as vectors.

SOLVED PROBLEMS

(1) Suppose the position in two-dimensional space of a particle, as a function of time, is given by the function

$$\vec{\mathbf{s}}(t) = \begin{bmatrix} t^2 \\ t^3 \end{bmatrix}$$

What will be the velocity vector for this particle?

Solution

Take derivative of each component w r t time, you get x' = 2t and y'= $3t^2$. The velocity vector will be:

$$s'(t) = \begin{bmatrix} 2t \\ 3t^2 \end{bmatrix}$$

For the above example find the value of position and velocity at t= 3.

Solution

Simply substitute t =3 in the expression for s(t) and s'(t) to get s(3) = $\begin{bmatrix} t^2 \\ t^3 \end{bmatrix} = \begin{bmatrix} 9 \\ 27 \end{bmatrix}$

s'(3) = $\begin{bmatrix} 2t \\ 3t^2 \end{bmatrix} = \begin{bmatrix} 6 \\ 27 \end{bmatrix}$

SELF-TEST 10

(1) The derivative of a vector with respect to a scalar quantity is

(A) a vector

(B) a scalar

(C) either a scalar or a vector

(D) a matrix

(2) The derivative of a vector which represents a position vector gives

(A) the magnitude of the position vector

(B) a square matrix whose diagonal elements are the given vector

(C) the velocity vector

(D) the acceleration vector

SHORT ANSWER QUESTIONS 02

(1)Show that the dot product of two vectors **u** and **v** can be interpreted as the magnitude of **u** times the component of **v** in the direction of **u**.

(2)The work done by a force, represented by a vector F, in moving an object a given distance is the product of the component of force in the given direction times the distance moved. If the vector s represents the direction and magnitude (distance) the object is moved, show that the work done is equivalent to F.s .

(3)Prove that the dot product is commutative, **a . b = b . a** .

(4). Show, geometrically, that the dot and cross in the triple scalar product can be interchanged: $\boldsymbol{a}.(\boldsymbol{b} \times \boldsymbol{c}) = (\boldsymbol{a} \times \boldsymbol{b}).\boldsymbol{c}$.

(6). Show that the triple vector product $\boldsymbol{a}.(\boldsymbol{b} \times \boldsymbol{c})$ lies in the plane spanned by the vectors **a** and **b**.

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Course Companion Website

Visit Here for Course Companion website for this course:

UNIT 02-02: MOMENTUM AND ENERGY

LEARNING OBJECTIVES

After successful completion of this unit, you will be able to

- Define linear momentum.
- Explain the relationship between momentum and force.
- State Newton’s second law of motion in terms of momentum.
- Calculate momentum, given mass and velocity.
- Describe the principle of conservation of momentum.
- Explain conservation of momentum with examples.
- Explain the concept of energy and conservation of energy
- Derive Work Energy principle for a particle in 1-D.

INTRODUCTION

Fig 02-01: Each of the players has great momentum (credit: ozzzie,Flickr)

We use the term momentum in various ways in everyday language, and most of these ways are consistent with its precise scientific definition. We speak of sports teams or politicians gaining and maintaining the momentum to win. We also recognize that momentum has something to do with collisions. For example, looking at the rugby players in the photograph colliding and falling to the ground, we expect their momenta to have great effects in the resulting collisions. Generally, momentum implies a tendency to continue on course—to move in the same direction—and is associated with great mass and speed.

Momentum, like energy, is important because it is conserved. Only a few physical quantities are conserved in nature, and studying them yields fundamental insight into how nature works, as we shall see in our study of momentum and energy.

Energy plays an essential role both in everyday events and in scientific phenomena. You can no doubt name many forms of energy, from that provided by our foods, to the energy we use to run our cars, to the sunlight that warms us on the beach. You can also cite examples of what people call energy that may not be scientific, such as someone having an energetic personality.

Not only does energy have many interesting forms, it is involved in almost all phenomena, and is one of the most important concepts of physics. What makes it even more important is that the total amount of energy in the universe is constant. Energy can change forms, but it cannot appear from nothing or disappear without a trace. Energy is thus one of a handful of physical quantities that we say is conserved.

Conservation of energy (as physicists like to call the principle that energy can neither be created nor destroyed) is based on experiment. Even as scientists discovered new forms of energy, conservation of energy has always been found to apply. Perhaps the most dramatic example of this was supplied by Einstein when he suggested that mass is equivalent to energy (his famous equation $E = mc^2$).

From a societal viewpoint, energy is one of the major building blocks of modern civilization. Energy resources are key limiting factors to economic growth. The world use of energy resources, especially oil, continues to grow, with ominous consequences economically, socially, politically, and environmentally.

There is no simple, yet accurate, scientific definition for energy. Energy is characterized by its many forms and the fact that it is conserved. We can loosely define energy as the ability to do work, admitting that in some circumstances not all energy is available to do work. Work is intimately related to energy and how energy moves from one system to another or changes form.

02-01: Conservation of Momentum

Linear Momentum

In Newtonian mechanics, linear momentum, translational momentum, or simply momentum is the product of the mass and velocity of an object. It is a vector quantity, possessing magnitude and direction in three-dimensional space. If m is an object's mass and v is the velocity (also a vector), then the momentum is

$$\mathbf{p} = m\,\mathbf{v},$$

In SI units, it is measured in kilogram meters per second (kg·m/s). Newton's second law of motion states that a body's rate of change in momentum is equal to the net force acting on it.

Momentum depends on the frame of reference, but in any inertial frame it is a conserved quantity, meaning that if a closed system is not affected by external forces, its total linear momentum does not change. Momentum is also conserved in special relativity, (with a modified formula) and, in a modified form, in electrodynamics, quantum mechanics, quantum field theory, and general relativity. It is an expression of one of the fundamental symmetries of space and time: translational symmetry.

Advanced formulations of classical mechanics, Lagrangian and Hamiltonian mechanics, allow one to choose coordinate systems that incorporate symmetries and constraints. In these systems the conserved quantity is generalized momentum, and in general this is different from the kinetic momentum defined above. The concept of generalized momentum is carried over into quantum mechanics, where it becomes an operator on a wave function. The momentum and position operators are related by the Heisenberg uncertainty principle.

In continuous systems such as electromagnetic fields, fluids and deformable bodies, a momentum density can be defined, and a continuum version of the conservation of momentum leads to equations such as the Navier–Stokes equations for fluids or the Cauchy momentum equation for deformable solids or fluids..

Newtonian Mechanics

Momentum is a vector quantity: it has both magnitude and direction. Since momentum has direction, it can be used to predict the resulting direction and speed of motion of objects after they collide. Below, the basic properties of momentum are described in one dimension. The vector equations are almost identical to the scalar equations (see multiple dimensions).

Single particle

The momentum of a particle is conventionally represented by the letter *p*. It is the product of two quantities, the particle's mass, m and its velocity v.

$$\mathbf{p} = m\,\mathbf{v}$$

The unit of momentum is the product of the units of mass and velocity. In SI units, if the mass is in kilograms and the velocity is in meters per second then the momentum is in kilogram meters per second (kg·m/s). In cgs units, if the mass is in grams and the velocity in centimeters per second, then the momentum is in gram centimeters per second (g·cm/s).

Being a vector, momentum has magnitude and direction. For example, a 1 kg model airplane, traveling due north at 1 m/s in straight and level flight, has a momentum of 1 kg·m/s due north measured with reference to the ground.

SOLVED PROBLEMS

EXAMPLE (1) CALCULATING MOMENTUM: A FOOTBALL PLAYER AND A FOOTBALL

(a) Calculate the momentum of a 110-kg football player running at 8.00 m/s. (b) Compare the player's momentum with the momentum of a hard-thrown 0.410-kg football that has a speed of 25.0 m/s.

Strategy

No information is given regarding direction, and so we can calculate only the magnitude of the momentum, p. In both parts of this example, the magnitude of momentum can be calculated directly from the definition of momentum given in the equation, which becomes

$p = mv$

when only magnitudes are considered.

Solution for (a)

To determine the momentum of the player, substitute the known values for the player's mass and speed into the equation.

$p_{player} = (110 \text{ kg})\ (8.00 \text{ m/s}) = 880 \text{ kg} \cdot \text{m/s}$

Solution for (b)

To determine the momentum of the ball, substitute the known values for the ball's mass and speed into the equation.

$p_{ball} = (0.410 \text{ kg})(25.0 \text{ m/s}) = 10.3 \text{ kg} \cdot \text{m/s}$

The ratio of the player's momentum to that of the ball is

$$\frac{p_{player}}{p_{ball}} = \frac{880}{10.3} = 85.9.$$

Discussion

Although the ball has greater velocity, the player has a much greater mass. Thus the momentum of the player is much greater than the momentum of the football, as you might guess. As a result, the player's motion is only slightly affected if he catches the ball. We shall quantify what happens in such collisions in terms of momentum in later sections.

Many particles

The momentum of a system of particles is the sum of their momenta (plural of momentum). If two particles have respective masses m_1 and m_2, and velocities v_1 and v_2, the total momentum is

$$\begin{aligned} p &= p_1 + p_2 \\ &= m_1 v_1 + m_2 v_2 \end{aligned}$$

(02-02-01)

The momenta of more than two particles can be added more generally with the following:

$$p = \sum_i m_i v_i$$

(02-02-02)

A system of particles has a center of mass, a point determined by the weighted sum of their positions:

$$r_{\text{cm}} = \frac{m_1 r_1 + m_2 r_2 + \cdots}{m_1 + m_2 + \cdots} = \frac{\sum_i m_i r_i}{\sum_i m_i}.$$

(02-02-03)

If all the particles are moving, the center of mass will generally be moving as well (unless the system is in pure rotation around it). If the center of mass is moving at velocity v_{cm}, the momentum is:

$$p = m v_{\text{cm}}.$$

(02-02-04)

This is known as Euler's first law.

Relation to force

If the net force applied to a particle is a constant F, and is applied for a time interval Δt, the momentum of the particle changes by an amount

$$\Delta p = F\Delta t\,.$$

(02-02-05)

In differential form, this is Newton's second law; the rate of change of the momentum of a particle is equal to the instantaneous force F acting on it,

$$F = \frac{dp}{dt}.$$

(02-02-06)

If the net force experienced by a particle changes as a function of time, F(t), the change in momentum (or impulse J) between times t_1 and t_2 is

$$\Delta p = J = \int_{t_1}^{t_2} F(t)\,dt\,.$$

(02-02-07)

Impulse is measured in the derived units of the newton second (1 N·s = 1 kg·m/s) or dyne second (1 dyne·s = 1 g·m/s)

Under the assumption of constant mass m, it is equivalent to write

$$F = \frac{d(mv)}{dt} = m\frac{dv}{dt} = ma,$$

(02-02-08)

hence the net force is equal to the mass of the particle, times its acceleration.

Solved Problems

Example (1): A model airplane of mass 1 kg accelerates from rest to a velocity of 6 m/s due north in 2 s. The net force required to produce this acceleration is 3 newton due north. The change in momentum is 6 kg·m/s. The rate of change of momentum is 3 (kg·m/s)/s = 3 N.

Example (2) Calculating Force: Venus Williams' Racquet

During the 2007 French Open, Venus Williams hit the fastest recorded serve in a premier women's match, reaching a speed of 58 m/s (209 km/h). What is the average force exerted on the 0.057-kg tennis ball by Venus Williams' racquet, assuming that the ball's speed just after impact is 58 m/s, that the initial horizontal component of the velocity before impact is negligible, and that the ball remained in contact with the racquet for 5.0 ms (milliseconds)?

Strategy

This problem involves only one dimension because the ball starts from having no horizontal velocity component before impact. Newton's second law stated in terms of momentum is then written as

$$\mathbf{F}_{\text{net}} = \frac{\Delta \mathbf{p}}{\Delta t}.$$

As mass is constant, the change in momentum is given by

$\Delta p = m\Delta v = m(v_f - v_i)$.

In this example, the velocity just after impact and the change in time are given; thus, once Δp is calculated, $F_{net} = \Delta p/\Delta t$ can be used to find the force.

Solution

To determine the change in momentum, substitute the values for the initial and final velocities into the equation above.

$$\begin{aligned} \Delta p &= m(v_f - v_i) \\ &= (0.057 \text{ kg})(58 \text{ m/s} - 0 \text{ m/s}) \\ &= 3.306 \text{ kg} \cdot \text{m/s} \approx 3.3 \text{ kg} \cdot \text{m/s} \end{aligned}$$

Now the magnitude of the net external force can determined by using $F_{net} = \Delta p/\Delta t$:

$$\begin{aligned} F_{\text{net}} &= \frac{\Delta p}{\Delta t} = \frac{3.306 \text{ kg} \cdot \text{m/s}}{5.0 \times 10^{-3} \text{ s}} \\ &= 661 \text{ N} \approx 660 \text{ N}, \end{aligned}$$

where we have retained only two significant figures in the final step.

Discussion

This quantity was the average force exerted by Venus Williams' racquet on the tennis ball during its brief impact (note that the ball also experienced the 0.56 N force of gravity, but that force was not due to the racquet). This problem could also be solved by first finding the acceleration and then using $F_{net} = ma$, but one additional step would be required compared with the strategy used in this example.

Conservation of Linear Momentum

In a closed system (one that does not exchange any matter with its surroundings and is not acted on by external forces) the total momentum is constant. This fact, known as the law of conservation of momentum, is implied by Newton's laws of motion. Suppose, for example, two particles interact. According to Newton's third law of motion, the forces between them are equal and opposite. If the particles are numbered 1 and 2, the second law states that $F_1 = dp_1/dt$ and $F_2 = dp_2/dt$. Therefore,

$$\frac{dp_1}{dt} = -\frac{dp_2}{dt},$$

(02-02-09)

with the negative sign indicating that the forces oppose. Equivalently,

$$\frac{d}{dt}(p_1 + p_2) = 0.$$

(02-02-10)

If the velocities of the particles are u_1 and u_2 before the interaction, and afterwards they are v_1 and v_2, then

$$m_1 u_1 + m_2 u_2 = m_1 v_1 + m_2 v_2.$$

(02-02-11)

This law holds good, no matter how complicated the force is between particles. Similarly, if there are several particles, the momentum exchanged between each pair of particles adds up to zero, so the total change in momentum is zero. This conservation law applies to all interactions, including collisions and separations caused by explosive forces. It can also be generalized to situations where Newton's laws do not hold, for example in the theory of relativity and in electrodynamics.

Elastic Collisions in One Dimension

Let us consider various types of two-object collisions. These collisions are the easiest to analyze, and they illustrate many of the physical principles involved in collisions. The conservation of momentum principle is very useful here, and it can be used whenever the net external force on a system is zero.

We start with the elastic collision of two objects moving along the same line—a one-dimensional problem. An elastic collision is one that also conserves internal kinetic energy. Internal kinetic energy is the sum of the kinetic energies of the objects in the system. Figure 02-02 illustrates an elastic collision in which internal kinetic energy and momentum are conserved.

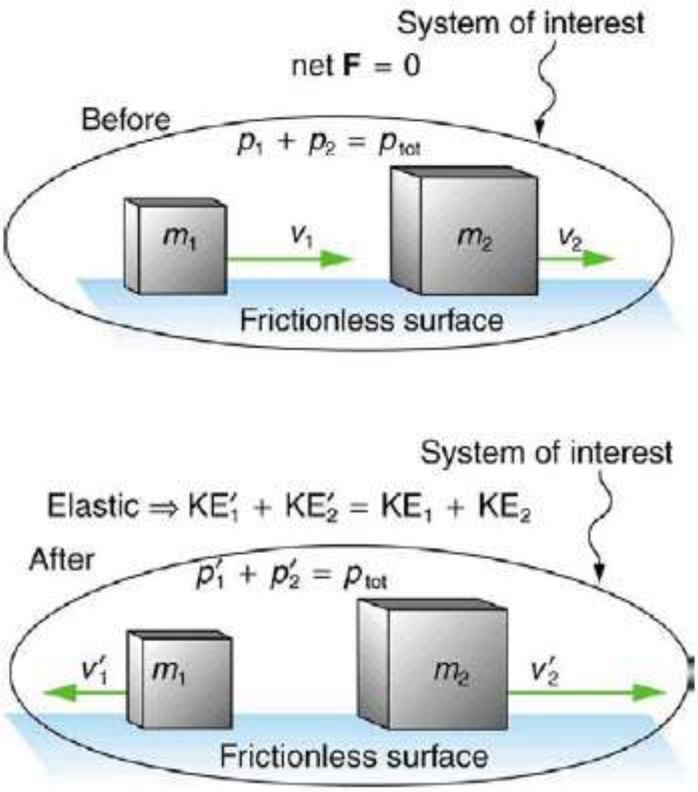

Fig 02-02: An elastic one-dimensional two-object collision. Momentum and internal kinetic energy are conserved.

Truly elastic collisions can only be achieved with subatomic particles, such as electrons striking nuclei. Macroscopic collisions can be very nearly, but not quite, elastic—some kinetic energy is always converted into other forms of energy such as heat transfer due to friction and sound. One macroscopic collision that is nearly elastic is that of two steel blocks on ice. Another nearly elastic collision is that between two carts with spring bumpers on an air track. Icy surfaces and air tracks are nearly frictionless, more readily allowing nearly elastic collisions on them.

Now, to solve problems involving one-dimensional elastic collisions between two objects we can use the equations for conservation of momentum and conservation of internal kinetic energy. First, the equation for conservation of momentum for two objects in a one-dimensional collision is

$$p_1 + p_2 = p'_1 + p'_2 \quad (F_{net} = 0)$$

or

$$m_1 v_1 + m_2 v_2 = m_1 v'_1 + m_2 v'_2 \quad (F_{net} = 0),$$

where the primes (') indicate values after the collision. By definition, an elastic collision conserves internal kinetic energy, and so the sum of kinetic energies before the collision equals the sum after the collision. Thus,

$$\frac{1}{2}m_1 v_1^2 + \frac{1}{2}m_2 v_2^2 = \frac{1}{2}m_1 v'_1{}^2 + \frac{1}{2}m_2 v'_2{}^2 \quad \text{(two-object elastic collision)}$$

gives the equation for conservation of internal kinetic energy in a one-dimensional collision.

Solved Problems

Calculate the velocities of two objects following an elastic collision, given that

$m_1 = 0.500$ kg, $m_2 = 3.50$ kg, $v_1 = 4.00$ m/s, and $v_2 = 0$.

Strategy and Concept

First, visualize what the initial conditions mean—a small object strikes a larger object that is initially at rest. This situation is slightly simpler than the situation where both objects are initially moving. We are asked to find two unknowns (the final velocities v'_1 and v'_2). To find two unknowns, we must use two independent equations. Because this collision is elastic, we can use the above two equations. Both can be simplified by the fact that object 2 is initially at rest, and thus $v_2 = 0$. Once we simplify these equations, we combine them algebraically to solve for the unknowns.

Solution

For this problem, note that $v_2 = 0$ and use conservation of momentum. Thus,

$p_1 = p'_1 + p'_2$

or

$m_1 v_1 = m_1 v'_1 + m_2 v'_2$.

Using conservation of internal kinetic energy and that $v_2 = 0$,

$$\frac{1}{2}m_1 v_1^2 = \frac{1}{2}m_1 v'_1{}^2 + \frac{1}{2}m_2 v'_2{}^2.$$

Solving the first equation (momentum equation) for v'_2, we obtain

$$v'_2 = \frac{m_1}{m_2}(v_1 - v'_1).$$

Substituting this expression into the second equation (internal kinetic energy equation) eliminates the variable v'_2, leaving only v'_1 as an unknown (the algebra is left as an exercise for you). There are two solutions to any quadratic equation; in this example, they are $v'_1 = 4.00$ m/s and $v'_1 = -3.00$ m/s.

As noted when quadratic equations were encountered in earlier chapters, both solutions may or may not be meaningful. In this case, the first solution is the same as the initial condition. The first solution thus represents the situation before the collision and is discarded. The second solution ($v'_1 = -3.00$ m/s) is negative, meaning that the first object bounces backward. When this negative value of v'_1 is used to find the velocity of the second object after the collision, we get

$$v'_2 = \frac{m_1}{m_2}(v_1 - v'_1) = \frac{0.500 \text{ kg}}{3.50 \text{ kg}}[4.00 - (-3.00)] \text{ m/s}$$

or

$$v'_2 = 1.00 \text{ m/s}.$$

Discussion

The result of this example is intuitively reasonable. A small object strikes a larger one at rest and bounces backward. The larger one is knocked forward, but with a low speed. (This is like a compact car bouncing backward off a full-size SUV that is initially at rest.) As a check, try calculating the internal kinetic energy before and after the collision. You will see that the internal kinetic energy is unchanged at 4.00 J. Also check the total momentum before and after the collision. You will find it unchanged.

The equations for conservation of momentum and internal kinetic energy as written above can be used to describe any one dimensional elastic collision of two objects. These equations can be extended to more objects if needed.

DEPENDENCE ON REFERENCE FRAME

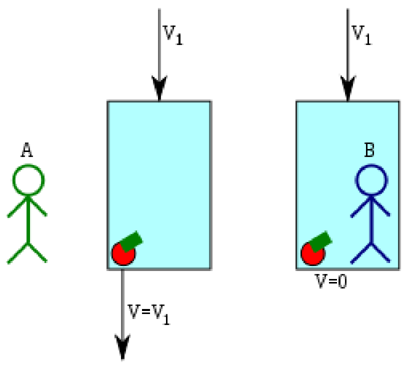

Fig 02-03: Newton's apple in Einstein's elevator. In person A's frame of reference, the apple has non-zero velocity and momentum. In the elevator's and person B's frames of reference, it has zero velocity and momentum.

Momentum is a measurable quantity, and the measurement depends on the motion of the observer. For example: if an apple is placed in a glass elevator that is descending, an outside observer, looking into the elevator, sees the apple moving, so, to that observer, the apple has a non-zero momentum. To someone inside the elevator, the apple does not move, so, it has zero momentum. The two observers have different frame of reference, in which, they observe motions, and, if the elevator is descending steadily, they will see behavior that is consistent with those same physical laws.

Suppose a particle has position x in a stationary frame of reference. From the point of view of another frame of reference, moving at a uniform speed u, the position (represented by a primed coordinate) changes with time as

$$x' = x - ut\,.$$

(02-02-12)

This is called a Galilean transformation. If the particle is moving at speed dx/dt = v in the first frame of reference, in the second, it is moving at speed

$$v' = \frac{dx'}{dt} = v - u\,.$$

(02-02-13)

Since u does not change, the accelerations are the same:

$$a' = \frac{dv'}{dt} = a\,.$$

(02-02-14)

Thus, momentum is conserved in both reference frames. Moreover, as long as the force has the same form, in both frames, Newton's second law is unchanged. Forces such as Newtonian gravity, which depend only on the scalar distance between objects, satisfy this criterion. This independence of reference frame is called Newtonian relativity or Galilean invariance.

A change of reference frame, can, often, simplify calculations of motion. For example, in a collision of two particles, a reference frame can be chosen, where, one particle begins at rest. Another, commonly used reference frame is the center of mass frame – one that is moving with the center of mass. In this frame, the total momentum is zero.

02-02: Work and Energy

Work

In physics, a force is said to do work if, when acting, there is a displacement of the point of application in the direction of the force. For example, when a ball is held above the ground and then dropped, the work done on the ball as it falls is equal to the weight of the ball (a force) multiplied by the distance to the ground (a displacement).

Work transfers energy from one place to another or one form to another.

According to Jammer, the term work was introduced in 1826 by the French mathematician Gaspard-Gustave Coriolis as "weight lifted through a height", which is based on the use of early steam engines to lift buckets of water out of flooded ore mines. According to Dugas, it is to Solomon of Caux "that we owe the term work in the sense that it is used in mechanics now".

Units

The SI unit of work is the joule (J), which is defined as the work expended by a force of one newton through a displacement of one meter.

The dimensionally equivalent newton-metre (N·m) is sometimes used as the measuring unit for work, but this can be confused with the unit newton.metre, which is the measurement unit of torque. Usage of N·m is discouraged by the SI authority, since it can lead to confusion as to whether the quantity expressed in newton.metre is a torque measurement, or a measurement of work.

Non-SI units of work include newton.metre, erg, foot-pound, foot-poundal, kilowatt hour, the litre-atmosphere, and the horsepower-hour. Due to work having the same physical dimension as heat, occasionally measurement units typically reserved for heat or energy content, such as therm, BTU and Calorie, are utilized as a measuring unit.

Work and Energy

The work W done by a constant force of magnitude F on a point that undergoes a displacement s in a straight line in the direction of the force is the product of F and s

$$W = F.s$$

For example, if a force of 10 newton ($F = 10$ N) acts along a point that travels 2 metres ($s = 2$ m), then $W = F s = (10 \ \mathrm{N})(2 \ \mathrm{m}) = 20 \ \mathrm{J}$.

This is approximately the work done in lifting a 1 kg object from ground level to over a person's head against the force of gravity. Notice that the work is doubled either by lifting twice the weight the same distance or by lifting the same weight twice the distance.

Work is closely related to energy. The work-energy principle states that an increase in the kinetic energy of a rigid body is caused by an equal amount of positive work done on the body by the resultant force acting on that body. Conversely, a decrease in kinetic energy is caused by an equal amount of negative work done by the resultant force.

From Newton's second law, it can be shown that work on a free (no fields), rigid (no internal degrees of freedom) body, is equal to the change in kinetic energy KE of the linear velocity and angular velocity of that body,

$$W = \Delta KE .$$

(02-02-15)

The work of forces generated by a potential function is known as potential energy and the forces are said to be conservative. Therefore, work on an object that is merely displaced in a conservative force field, without change in velocity or rotation, is equal to minus the change of potential energy P E of the object,

$$W = -\Delta PE .$$

(02-02-16)

These formulas show that work is the energy associated with the action of a force, so work subsequently possesses the physical dimensions, and units, of energy. The work/energy principles discussed here are identical to Electric work/energy principles.

MATHEMATICAL CALCULATION

For moving objects, the quantity of work/time (power) is integrated along the trajectory of the point of application of the force. Thus, at any instant, the rate of the work done by a force (measured in joules/second, or watts) is the scalar product of the force (a vector), and the velocity vector of the point of application. This scalar product of force and velocity is known as instantaneous power. Just as velocities may be integrated over time to obtain a total distance, by the fundamental theorem of calculus, the total work along a path is similarly the time-integral of instantaneous power applied along the trajectory of the point of application.

Work is the result of a force on a point that follows a curve X, with a velocity v, at each instant. The small amount of work δW that occurs over an instant of time dt is calculated as

$$\delta W = \mathbf{F} \cdot d\mathbf{s} = \mathbf{F} \cdot \mathbf{v}dt$$

where F · v is the power over the instant dt. The sum of these small amounts of work over the trajectory of the point yields the work,

$$W = \int_{t_1}^{t_2} \mathbf{F} \cdot \mathbf{v}dt = \int_{t_1}^{t_2} \mathbf{F} \cdot \tfrac{d\mathbf{s}}{dt}dt = \int_C \mathbf{F} \cdot d\mathbf{s},$$

where C is the trajectory from $x(t_1)$ to $x(t_2)$. This integral is computed along the trajectory of the particle, and is therefore said to be path dependent.

If the force is always directed along this line, and the magnitude of the force is F, then this integral simplifies to

$$W = \int_C F\,ds$$

where s is displacement along the line. If F is constant, in addition to being directed along the line, then the integral simplifies further to

$$W = \int_C F\,ds = F\int_C ds = Fs$$

where s is the displacement of the point along the line.

This calculation can be generalized for a constant force that is not directed along the line, followed by the particle. In this case the dot product F · ds = F cos θ ds, where θ is the angle between the force vector and the direction of movement, that is

$$W = \int_C \mathbf{F} \cdot d\mathbf{s} = Fs\cos\theta.$$

In the notable case of a force applied to a body always at an angle of 90° from the velocity vector (as when a body moves in a circle under a central force), no work is done at all, since the cosine of 90 degrees is zero. Thus, no work can be performed by gravity on a planet with a circular orbit (this is ideal, as all orbits are slightly elliptical). Also, no work is done on a body moving circularly at a constant speed while constrained by mechanical force, such as moving at constant speed in a frictionless ideal centrifuge.

Solved Problems

Example (1) Calculating the Work You Do to Push a Lawn Mower Across a Large Lawn

How much work is done on the lawn mower by the person in Fig 02-04 if he exerts a constant force of 75.0 N at an angle 35º below the horizontal and pushes the mower 25.0 m on level ground? Convert the amount of work from joules to kilocalories and compare it with this person's average daily intake of 10,000 kJ (about 2400 kcal) of food energy. One calorie (1 cal) of heat is the amount required to warm 1 g of water by 1ºC , and is equivalent to 4.184 J , while one food calorie (1 kcal) is equivalent to 4184 J .

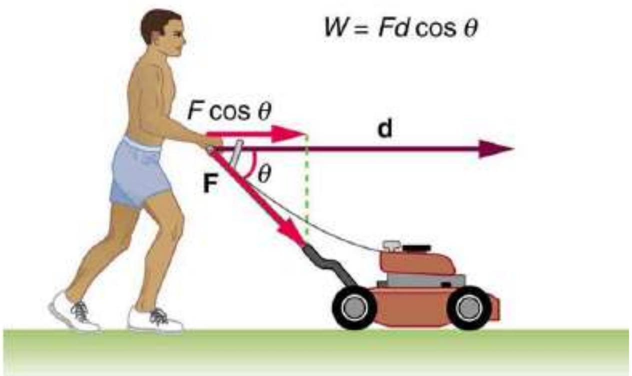

Fig 02-04: A person does work while mowing a lawn

Strategy

We can solve this problem by substituting the given values into the definition of work done on a system, stated in the equation W = Fd cosθ. The force, angle, and displacement are given, so that only the work W is unknown.

Solution

The equation for the work is

$W = Fd \cos \theta$.

Substituting the known values gives

$W = (75.0\ \text{N})(25.0\ \text{m}) \cos (35.0^\circ)$

$= 1536\ \text{J} = 1.54 \times 10^3\ \text{J}$.

Converting the work in joules to kilocalories yields $W = (1536\ \text{J})(1\ \text{kcal} / 4184\ \text{J}) = 0.367\ \text{kcal}$. The ratio of the work done to the daily consumption is

$$\frac{W}{2400\ \text{kcal}} = 1.53\times10^{-4}.$$

Discussion

This ratio is a tiny fraction of what the person consumes, but it is typical. Very little of the energy released in the consumption of food is used to do work. Even when we "work" all day long, less than 10% of our food energy intake is used to do work and more than 90% is converted to thermal energy or stored as chemical energy in fat.

WORK DONE BY A VARIABLE FORCE

Calculating the work as "force times the straight path segment" would only apply in the most simple of circumstances, as noted above. If force is changing, or if the body (not necessarily rigid) is moving along a curved path (for example due to rotation), then only the path of the application point of the force is relevant for the work done. Here only the component of the force parallel to the application point velocity is doing work (positive work when in the same direction, and negative when in the opposite direction of the velocity). This component of force can be described by the scalar quantity called scalar tangential component F.cos θ, (where θ is the angle between the force and the velocity).

And then the most general definition of work can be formulated as follows:

Work of a force is the line integral of its scalar tangential component along the path of its application point.

If the force varies (e.g. compressing a spring) we need to use calculus to find the work done. If the force is given by F(x) (a function of x) then the work done by the force along the x-axis from a to b is:

$$W = \int_a^b \mathbf{F}(\mathbf{s}) \cdot d\mathbf{s}$$

WORK AND POTENTIAL ENERGY

The scalar product of a force F and the velocity v of its point of application defines the power input to a system at an instant of time. Integration of this power over the trajectory of the point of application, C = x(t), defines the work input to the system by the force.

PATH DEPENDENCE

Therefore, the work done by a force F on an object that travels along a curve C is given by the line integral:

$$W = \int_C \mathbf{F} \cdot d\mathbf{x} = \int_{t_1}^{t_2} \mathbf{F} \cdot \mathbf{v} dt,$$

where dx(t) defines the trajectory C and v is the velocity along this trajectory. In general this integral requires the path along which the velocity is defined, so the evaluation of work is said to be path dependent.

The time derivative of the integral for work yields the instantaneous power,

$$\frac{dW}{dt} = P(t) = \mathbf{F} \cdot \mathbf{v}.$$

PATH INDEPENDENCE

If the work for an applied force is independent of the path, then the work done by the force, by the gradient theorem, defines a potential function which is evaluated at the start and end of the trajectory of the point of application. This means that there is a potential function U(x), that can be evaluated at the two points $x(t_1)$ and $x(t_2)$ to obtain the work over any trajectory between these two points. It is tradition to define this function with a negative sign so that positive work is a reduction in the potential, that is

$$W = \int_C \mathbf{F} \cdot d\mathbf{x} = \int_{\mathbf{x}(t_1)}^{\mathbf{x}(t_2)} \mathbf{F} \cdot d\mathbf{x} = U(\mathbf{x}(t_1)) - U(\mathbf{x}(t_2)).$$

The function U(x) is called the potential energy associated with the applied force. The force derived from such a potential function is said to be conservative. Examples of forces that have potential energies are gravity and spring forces.

In this case, the gradient of work yields

$$\nabla W = -\nabla U = -(\partial U \partial x, \partial U \partial y, \partial U \partial z) = F,$$

and the force F is said to be "derivable from a potential."

Because the potential U defines a force F at every point x in space, the set of forces is called a force field. The power applied to a body by a force field is obtained from the gradient of the work, or potential, in the direction of the velocity V of the body, that is

$$P(t) = -\nabla U \cdot v = F \cdot v.$$

WORK BY GRAVITY

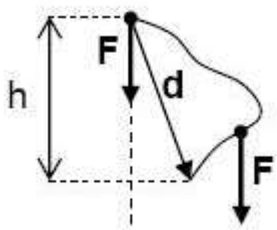

Fig 02-05: Gravity F = mg does work W = mgh along any descending path

In the absence of other forces, gravity results in a constant downward acceleration of every freely moving object. Near Earth's surface the acceleration due to gravity is g = 9.8 m·s^{-2} and the gravitational force on an object of mass m is F = mg. It is convenient to imagine this gravitational force concentrated at the center of mass of the object.

If an object is displaced upwards or downwards a vertical distance $y_2 - y_1$, the work W done on the object by its weight mg is:

$$W = F_g(y_2 - y_1) = F_g \Delta y = -mg\Delta y$$

where F_g is weight (pounds in imperial units, and newton in SI units), and Δy is the change in height y. Notice that the work done by gravity depends only on the vertical movement of the object. The presence of friction does not affect the work done on the object by its weight.

WORK–ENERGY PRINCIPLE

The principle of work and kinetic energy (also known as the work–energy principle) states that the work done by all forces acting on a particle (the work of the resultant force) equals the change in the kinetic energy of the particle. That is, the work W done by the resultant force on a particle equals the change in the particle's kinetic energy E_k

$$W = \Delta E_k = \tfrac{1}{2}mv_2^2 - \tfrac{1}{2}mv_1^2 ,$$

where v_1 and v_2 are the speeds of the particle before and after the work is done, and m is its mass.

The derivation of the work–energy principle begins with Newton’s second law of motion and the resultant force on a particle. Computation of the scalar product of the forces with the velocity of the particle evaluates the instantaneous power added to the system.

Constraints define the direction of movement of the particle by ensuring there is no component of velocity in the direction of the constraint force. This also means the constraint forces do not add to the instantaneous power. The time integral of this scalar equation yields work from the instantaneous power, and kinetic energy from the scalar product of velocity and acceleration. The fact that the work–energy principle eliminates the constraint forces underlies Lagrangian mechanics.

This section focuses on the work–energy principle as it applies to particle dynamics. In more general systems work can change the potential energy of a mechanical device, the thermal energy in a thermal system, or the electrical energy in an electrical device. Work transfers energy from one place to another or one form to another.

DERIVATION FOR A PARTICLE MOVING ALONG A STRAIGHT LINE

In the case the resultant force F is constant in both magnitude and direction, and parallel to the velocity of the particle, the particle is moving with constant acceleration along a straight line. The relation between the net force and the acceleration is given by the equation F = ma (Newton's second law), and the particle displacement s can be expressed by the equation

$$s = \frac{v_2^2 - v_1^2}{2a}$$

This follows from Equations of motion $v^2=u^2+2as$ which, in this case reduces to $v_2^2 = v_1^2 + 2as$.

The work of the net force is calculated as the product of its magnitude and the particle displacement. Substituting the above equations, one obtains:

$$W = Fs = mas = ma\left(\frac{v_2^2 - v_1^2}{2a}\right) = \frac{mv_2^2}{2} - \frac{mv_1^2}{2} = \Delta E_k$$

Solved Problems

Example (1) Calculating the Kinetic Energy of a Package

Suppose a 30.0-kg package on the roller belt conveyor system in Figure02-06 is moving at 0.500 m/s. What is its kinetic energy?

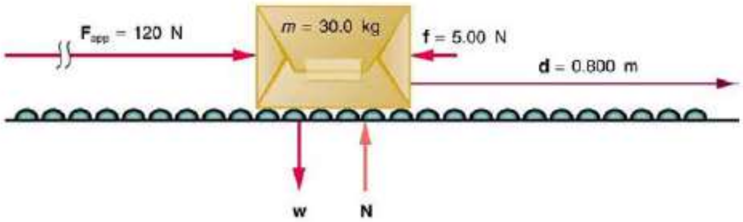

Fig 02-06: Kinetic energy of a package on roller

Strategy

Because the mass m and speed v are given, the kinetic energy can be calculated from its definition as given in the equation $K.E. = \frac{1}{2}mv^2$.

Solution

The kinetic energy is given by $K.E. = \frac{1}{2}mv^2$

Entering known values gives

KE = 0.5(30.0 kg)(0.500 m/s)^2,

which yields

KE = 3.75 kg · m^2/s^2 = 3.75 J.

Discussion

Note that the unit of kinetic energy is the joule, the same as the unit of work, as mentioned when work was first defined. It is also interesting that, although this is a fairly massive package, its kinetic energy is not large at this relatively low speed. This fact is consistent with the observation that people can move packages like this without exhausting themselves.

Example (2) Determining the Work to Accelerate a Package

Suppose that you push on the 30.0-kg package in Figure 02-06 with a constant force of 120 N through a distance of 0.800 m, and that the opposing friction force averages 5.00 N.

(a) Calculate the net work done on the package. (b) Solve the same problem as in part (a), this time by finding the work done by each force that contributes to the net force.

Strategy and Concept for (a)

This is a motion in one dimension problem, because the downward force (from the weight of the package) and the normal force have equal magnitude and opposite direction, so that they cancel in calculating the net force, while the applied force, friction, and the displacement are all horizontal. As expected, the net work is the net force times distance.

Solution for (a)

The net force is the push force minus friction, or F_{net} = 120 N – 5.00 N = 115 N. Thus the net work is $W_{net} = F_{net}\, d$ = (115 N)(0.800 m) = 92.0 N · m = 92.0 J.

Discussion for (a)

This value is the net work done on the package. The person actually does more work than this, because friction opposes the motion. Friction does negative work and removes some of the energy the person expends and converts it to thermal energy.

The net work equals the sum of the work done by each individual force.

Strategy and Concept for (b)

The forces acting on the package are gravity, the normal force, the force of friction, and the applied force. The normal force and force of gravity are each perpendicular to the displacement, and therefore do no work.

Solution for (b)

The applied force does work.

$$W_{app} = F_{app} d \cos(0^{o}) = F_{app} d$$

$$= (120\text{ N})(0.800\text{ m})$$

$$= 96.0\text{ J}$$

The friction force and displacement are in opposite directions, so that θ = 180º, and the work done by friction is

$$W_{fr} = F_{fr} d \cos(180^{o}) = -F_{fr} d$$

$$= -(5.00\text{ N})(0.800\text{ m})$$

$$= -4.00\text{ J.}$$

So the amounts of work done by gravity, by the normal force, by the applied force, and by friction are, respectively,

$$W_{gr} = 0,$$

$W_N = 0$,

$W_{app} = 96.0$ J,

$W_{fr} = -4.00$ J.

The total work done as the sum of the work done by each force is then seen to be

$W_{total} = W_{gr} + W_N + W_{app} + W_{fr} = 92.0$ J.

Discussion for (b)

The calculated total work W_{total} as the sum of the work by each force agrees, as expected, with the work W_{net} done by the net force. The work done by a collection of forces acting on an object can be calculated by either approach.

SELF-TEST 11

(1) The linear momentum is conserved

(A) for all systems

(B) for systems which are isolated

(C) only for systems which are in equilibrium

(D) for systems at rest only

(2) Momentum is a

(A) scalar quantity

(B) vector quantity

(C) matrix

(D) it may be scalar or vector

(3) The net momentum of a system of two particles will be

(A) dot product of the momentum of the two particles

(B) cross product of the momentum of two particles

(C) vector addition of the momenta for the two particles

(D) scalar addition of the magnitude of the momenta for the two particles

(4) Force is equal to

(A) impulse divided by the time of application of force

(B) impulse multipled by time of application of force

(C) impulse minus the time of application of force

(D) impulse

(5) During elastic collision

(i) total energy is conserved

(ii) internal kinetic energy is conserved

(iii) angular momentum is conserved

(iv) the linear momentum is conserved if no external force is involved

Choose the right option

(A) only(i) is true

(B) only (ii) is true

(C) option (i) and (iv) is true

(D) option (i), (ii) and (iv) are true

(6) A 4 ton (4000 kg) truck with velocity 40 kmph (11.1 m/s) collides head-on with a 1 ton (1000 kg) car moving with speed 100 kmph and both come to halt.

(A) the KE lost by car is more than that of the truck, hence car suffers more losses

(B) the force exerted by car on the truck is more, hence car suffers more losses

(C) the KE lost by the truck is more than the car, hence car suffers more losses

(D) the force exerted by truck on the car is more, hence car suffers more losses

(7) In conservative forces

(A) the work done by the object is independent of the path

(B) the work done by the object depends on the path

(C) the work done by the object may or may not depend on path

(D) the work done by the object is always zero

Short Answer Questions 02

(1) Explain the principle of conservation of momentum. Give an example in which the momentum is **not** conserved.

(2) What is the relation between momentum and impulse?

(3) How do you define Work and Energy?

(4) How can we calculate work done by a varying force F(x) as a function of space position x?

(5) Explain the path dependence and path independence of work by giving examples.

(6) What is the work energy principle and how can it be derived for a particle moving in 1 dimension (a straight line)

02-03: Conservation of Energy

Conservation of Energy

In physics, the law of conservation of energy states that the total energy of an isolated system remains constant, it is said to be conserved over time. This law means that energy can neither be created nor destroyed; rather, it can only be transformed or transferred from one form to another. For instance, chemical energy is converted to kinetic energy when a stick of dynamite explodes. If one adds up all the forms of energy that were released in the explosion, such as the kinetic energy of the pieces, as well as heat and sound, one will get the exact decrease of chemical energy in the combustion of the dynamite. Classically, conservation of energy was distinct from conservation of mass; however, special relativity showed that mass could be converted to energy and vice versa by $E = mc^2$, and science now takes the view that mass–energy is conserved.

Scientifically speaking, conservation of energy can be rigorously proven by Noether's theorem as a consequence of continuous time translation symmetry; that is, from the fact that the laws of physics do not change over time.

A consequence of the law of conservation of energy is that a perpetual motion machine of the first kind cannot exist, that is to say, no system without an external energy supply can deliver an unlimited amount of energy to its surroundings.

Historical Development

Ancient philosophers as far back as Thales of Miletus c. 550 BCE had inklings of the conservation of some underlying substance of which everything is made. However, there is no particular reason to identify this with what we know today as "mass-energy" (for example, Thales thought it was water). Empedocles (490–430 BCE) wrote that in this universal system, composed of four roots (earth, air, water, fire), "nothing comes to be or perishes"; instead, these elements suffer continual rearrangement.

In 1605, Simon Stevinus was able to solve a number of problems in statics based on the principle that perpetual motion was impossible.

In 1638, Galileo published his analysis of several situations—including the celebrated "interrupted pendulum"—which can be described (in modern language) as conservatively converting potential energy to kinetic energy and back again. Essentially, he pointed out that the height a moving body raises is equal to the height from which it falls, and used this observation to infer the idea of inertia. The remarkable aspect of this observation is that the height to which a moving body ascends on a frictionless surface does not depend on the shape of the surface.

In 1669, Christian Huygens published his laws of collision. Among the quantities he listed as being invariant before and after the collision of bodies was both the sum of their linear momentums as well as the sum of their kinetic energies. However, the difference between elastic and inelastic collision was not understood at the time. This led to the dispute among later researchers as to which of these conserved quantities was the more fundamental. In his Horologium Oscillatorium, he gave a much clearer statement regarding the height of ascent of a moving body, and connected this idea with the impossibility of a perpetual motion. Huygens' study of the dynamics of pendulum motion was based on a single principle: that the center of gravity of a heavy object cannot lift itself.

The fact that kinetic energy is scalar, unlike linear momentum which is a vector, and hence easier to work with, did not escape the attention of Gottfried Wilhelm Leibniz. It was Leibniz during 1676–1689 who first attempted a mathematical formulation of the kind of energy which is connected with motion (kinetic energy). Using Huygens' work on collision, Leibniz noticed that in many mechanical systems (of several masses, m_i each with velocity v_i),

$$\sum_i m_i v_i^2$$

was conserved so long as the masses did not interact. He called this quantity the vis viva or living force of the system. The principle represents an accurate statement of the approximate conservation of kinetic energy in situations where there is no friction. Many physicists at that time, such as Newton, held that the conservation of momentum, which holds even in systems with friction, as defined by the momentum:

$$\sum_i m_i v_i$$

was the conserved vis viva. It was later shown that both quantities are conserved simultaneously, given the proper conditions such as an elastic collision.

In 1687, Isaac Newton published his Principia, which was organized around the concept of force and momentum. However, the researchers were quick to recognize that the principles set out in the book, while fine for point masses, were not sufficient to tackle the motions of rigid and fluid bodies. Some other principles were also required.

The law of conservation of vis viva was championed by the father and son duo, Johann and Daniel Bernoulli. The former enunciated the principle of virtual work as used in statics in its full generality in 1715, while the latter based his Hydrodynamica, published in 1738, on this single conservation principle. Daniel's study of loss of vis viva of flowing water led him to formulate the Bernoulli's principle, which relates the loss to be proportional to the change in hydrodynamic pressure. Daniel also formulated the notion of work and efficiency for hydraulic machines; and he gave a kinetic theory of gases, and linked the kinetic energy of gas molecules with the temperature of the gas.

This focus on the vis viva by the continental physicists eventually led to the discovery of stationarity principles governing mechanics, such as the D'Alembert's principle, Lagrangian, and Hamiltonian formulations of mechanics.

Émilie du Châtelet (1706 – 1749) proposed and tested the hypothesis of the conservation of total energy, as distinct from momentum. Inspired by the theories of Gottfried Leibniz, she repeated and publicized an experiment originally devised by Willem's Gravesande in 1722 in which balls were dropped from different heights into a sheet of soft clay. Each ball's kinetic energy - as indicated by the quantity of material displaced - was shown to be proportional to the square of the velocity. The deformation of the clay was found to be directly proportional to the height the balls were dropped from, equal to the initial potential energy. Earlier workers, including Newton and Voltaire, had all believed that "energy" (so far as they understood the concept at all) was not distinct from momentum and therefore proportional to velocity. According to this understanding, the deformation of the clay should have been proportional to the square root of the height from which the balls were dropped from. In classical physics the correct formula is $E_k = 1/2\ mv^2$, where E_k is the kinetic energy of an object, m its mass and v its speed. On this basis, Châtelet proposed that energy must always have the same dimensions in any form, which is necessary to be able to relate it in different forms (kinetic, potential, heat…).

Engineers such as John Smeaton, Peter Ewart, Carl Holtzmann, Gustave-Adolphe Hirn and Marc Seguin recognized that conservation of momentum alone was not adequate for practical calculation and made use of Leibniz's principle. The principle was also championed by some chemists such as William Hyde Wollaston. Academics such as

John Playfair were quick to point out that kinetic energy is clearly not conserved. This is obvious to a modern analysis based on the second law of thermodynamics, but in the 18th and 19th centuries the fate of the lost energy was still unknown.

Gradually it came to be suspected that the heat inevitably generated by motion under friction was another form of vis viva. In 1783, Antoine Lavoisier and Pierre-Simon Laplace reviewed the two competing theories of vis viva and caloric theory. Count Rumford's 1798 observations of heat generation during the boring of cannons added more weight to the view that mechanical motion could be converted into heat, and (as importantly) that the conversion was quantitative and could be predicted (allowing for a universal conversion constant between kinetic energy and heat). Vis viva then started to be known as energy, after the term was first used in that sense by Thomas Young in 1807.

The recalibration of vis viva to

$$\frac{1}{2}\sum_i m_i v_i^2$$

which can be understood as converting kinetic energy to work, was largely the result of Gaspard-Gustave Coriolis and Jean-Victor Poncelet over the period 1819–1839. The former called the quantity quantité de travail (quantity of work) and the latter, travail mécanique (mechanical work), and both championed its use in engineering calculation.

In a paper Über die Natur der Wärme(German "On the Nature of Heat/Warmth"), published in the Zeitschrift für Physik in 1837, Karl Friedrich Mohr gave one of the earliest general statements of the doctrine of the conservation of energy in the words: "besides the 54 known chemical elements there is in the physical world one agent only, and this is called Kraft [energy or work]. It may appear, according to circumstances, as motion, chemical affinity, cohesion, electricity, light and magnetism; and from any one of these forms it can be transformed into any of the others."

Mechanical equivalent of heat

A key stage in the development of the modern conservation principle was the demonstration of the mechanical equivalent of heat. The caloric theory maintained that heat could neither be created nor destroyed, whereas conservation of energy entails the contrary principle that heat and mechanical work are interchangeable.

In the middle of the eighteenth century, Mikhail Lomonosov, a Russian scientist, postulated his corpusculo-kinetic theory of heat, which rejected the idea of a caloric.

Through the results of empirical studies, Lomonosov came to the conclusion that heat was not transferred through the particles of the caloric fluid.

In 1798, Count Rumford (Benjamin Thompson) performed measurements of the frictional heat generated in boring cannons, and developed the idea that heat is a form of kinetic energy; his measurements refuted caloric theory, but were imprecise enough to leave room for doubt.

The mechanical equivalence principle was first stated in its modern form by the German surgeon Julius Robert von Mayer in 1842. Mayer reached his conclusion on a voyage to the Dutch East Indies, where he found that his patients' blood was a deeper red because they were consuming less oxygen, and therefore less energy, to maintain their body temperature in the hotter climate. He discovered that heat and mechanical work were both forms of energy and in 1845, after improving his knowledge of physics; he published a monograph that stated a quantitative relationship between them.

Meanwhile, in 1843, James Prescott Joule independently discovered the mechanical equivalent in a series of experiments. In the most famous, now called the "Joule apparatus", a descending weight attached to a string caused a paddle immersed in water to rotate. He showed that the gravitational potential energy lost by the weight in descending was equal to the internal energy gained by the water through friction with the paddle.

Over the period 1840–1843, similar work was carried out by engineer Ludwig A. Colding, although it was little known outside his native Denmark.

Both Joule's and Mayer's work suffered from resistance and neglect but it was Joule's that eventually drew the wider recognition.

In 1844, William Robert Grove postulated a relationship between mechanics, heat, light, electricity and magnetism by treating them all as manifestations of a single "force" (energy in modern terms). In 1846, Grove published his theories in his book The Correlation of Physical Forces. In 1847, drawing on the earlier work of Joule, Sadi Carnot and Émile Clapeyron, Hermann von Helmholtz arrived at conclusions similar to Grove's and published his theories in his book Über die Erhaltung der Kraft (On the Conservation of Force, 1847). The general modern acceptance of the principle stems from this publication.

In 1850, William Rankine first used the phrase the law of the conservation of energy for the principle.

In 1877, Peter Guthrie Tait claimed that the principle originated with Sir Isaac Newton, based on a creative reading of propositions 40 and 41 of the Philosophiae Naturalis Principia Mathematica. This is now regarded as an example of Whig history.

MASS–ENERGY EQUIVALENCE

Matter is composed of such things as atoms, electrons, neutrons, and protons. It has intrinsic or rest mass. In the limited range of recognized experience of the nineteenth century it was found that such rest mass is conserved. Einstein's 1905 theory of special relativity showed that it corresponds to an equivalent amount of rest energy. This means that it can be converted to or from equivalent amounts of other (non-material) forms of energy, for example kinetic energy, potential energy, and electromagnetic radiant energy. When this happens, as recognized in twentieth century experience, rest mass is not conserved, unlike the total mass or total energy. All forms of energy contribute to the total mass and total energy.

For example, an electron and a positron each have rest mass. They can perish together, converting their combined rest energy into photons having electromagnetic radiant energy, but no rest mass. If this occurs within an isolated system that does not release the photons or their energy into the external surroundings, then neither the total mass nor the total energy of the system will change. The produced electromagnetic radiant energy contributes just as much to the inertia (and to any weight) of the system as did the rest mass of the electron and positron before their demise. Likewise, non-material forms of energy can perish into matter, which has rest mass.

Thus, conservation of energy (total, including material or rest energy), and conservation of mass (total, not just rest), each still holds as an (equivalent) law. In the 18th century these had appeared as two seemingly-distinct laws.

SELF-TEST 12

(1) Which of the following is the latest addition to the list of forms of energy, as known to us

(A) heat

(B) magnetism

(C) mass

(D) gravity

(2) Who established the equivalence of heat and mechanical energy?

(A) Newton

(B) Joule

(C) Einstein

(D) Stephan Hawking

(3) Who established the equivalence of mass and energy?

(A) Newton

(B) Joule

(C) Einstein

(D) Stephan Hawking

Short Answer Questions 03

(1) Explain the concept of conservation of energy.

(2) Elaborate the historical background of the concept of conservation of energy.

(3) Consider a simple pendulum. Prove that the energy is conserved for this system. (Ignore the effect of rotation of the Earth)

(4) Consider a body of mass m being thrown in upwards direction with an initial speed of u. Prove that the energy is conserved.

(5) Consider the experiment of open lift discussed in a Solved Problem "The elevator problem (one dimension accelerating systems)" being reproduced below.

"An open lift is moving with an acceleration a = 1.20 m/s^2 in upwards direction with respect to the person sitting on ground in Lab frame. The man in lift drops a ball out of the lift by releasing it when the lift was 4.00 m from ground. What will be the speed of a ball (just before it touches ground) in Lift's frame and in Lab's frame?"

(a) Find the total mechanical energy (KE+PE) for the ball with mass 0.5 kg in Lab frame as a function of time and check whether it is constant.

(b) Find the total mechanical energy (KE+PE) for the ball with mass 0.5 kg in Lift's frame as a function of time and check whether it is constant.

02-04: Motion of Rockets

In this section we consider moving bodies with variable mass. Such kind of motion often occurs in nature and technology. We can mention here for example:

- Falling of an evaporating raindrop;
- Movement of a melting ice block or an iceberg on the ocean surface;
- Movement of a squid or jellyfish;
- Rocket flight.

Below we derive a simple differential equation for the motion of body with variable mass considering as an example rocket motion.

Differential Equation of Rocket Motion

Rocket motion is based on Newton's third law, which states that "for every action there is an equal and opposite reaction". Hot gases are exausted through a nozzle of the rocket and produce the action force. The reaction force acting in the opposite direction is called the thrust force. The thrust force just causes the rocket acceleration.

Let the initial mass of the rocket be m and its initial velocity be v. In certain time dt, the mass of the rocket decreases by dm as a result of the fuel combustion. This leads the rocket velocity to be increased by dv. We apply the law of conservation of momentum to the system of the rocket and gas flow. At the initial moment the momentum of the system is equal to mv. In a small time dt the momentum of the rocket becomes

$$p_1=(m-dm)(v+dv),$$

(02-02-17)

and the momentum of the exhaust gases in the Earth's coordinate system is

$$p_2=dm(v-u),$$

(02-02-18)

where u is the exhaust gas velocity with respect to the Earth. Here we took into account that the exhaust velocity is in the opposite direction to the rocket movement (Figure). Therefore we have put the minus sign in front of u.

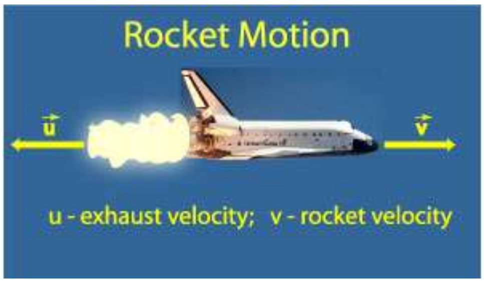

Figure 02-07: Rocket motion

By the law of conservation of the total momentum of the system, we can write:

$$p = p_1 + p_2, \quad \Rightarrow mv = (m - dm)(v + dv) + dm(v - u).$$

By transforming the given equation, we obtain:

$$\cancel{mv} = \cancel{mv} - \cancel{vdm} + mdv - dmdv + \cancel{vdm} - udm.$$

We can neglect the term dmdv in the last expression considering small increments of these values. As a result, the equation is written as

m.dv=u.dm.

We divide both sides by dt to convert the equation into the form of Newton's second law:

$$m\frac{dv}{dt} = u\frac{dm}{dt}.$$

The given equation is called the differential equation of rocket motion. The right side of the equation represents the thrust force T:

$$T = u\frac{dm}{dt}.$$

As it can be seen from the last formula, the thrust force is proportional to the exhaust velocity and the fuel burn rate.

Of course, the differential equation we derived describes an ideal case. It does not take into account the gravitational force or aerodynamic force. Their inclusion leads to significant complication of the differential equation.

IDEAL ROCKET EQUATION OR TSIOLKOVSKY ROCKET EQUATION

If we integrate the differential equation, we can get the dependence of the rocket velocity on the burned fuel mass. The resulting formula is called the ideal rocket equation or Tsiolkovsky rocket equation who derived it in 1897.

To get this formula it's convenient to use the differential equation in the form:

mdv=udm.

Separating the variables and integrating gives:

$$dv = u\frac{dm}{m}, \quad \Rightarrow \int_{v_0}^{v_1} dv = \int_{m_0}^{m_1} u\frac{dm}{m}.$$

Take into account that dm denotes mass decrease. Therefore we take the increment dm with the negative sign. As a result, the equation is written as follows:

$$v\Big|_{v_0}^{v_1} = -u\left(\ln m\right)\Big|_{m_0}^{m_1}, \quad \Rightarrow v_1 - v_0 = u\ln\frac{m_0}{m_1}.$$

where v_0 and v_1 are the initial and final velocities of the rocket, m_0 and m_1 are the initial and final masses of the rocket, respectively.

By setting $v_0=0$, we obtain the formula derived by Tsiolkovsky:

$$v = u\ln\frac{m_0}{m}.$$

This formula determines the rocket velocity depending on its mass change while the fuel is burning. It allows rough estimation of the fuel capacity necessary to accelerate the rocket to a given velocity

SOLVED PROBLEMS

Problem 02-07

Estimate the fuel mass needed to launch a small "nanosatellite" with mass of 50kg to a low orbit using a single-stage rocket. The specific impulse of the rocket is 3000ms^{-1}.

Solution

We can use the ideal rocket equation for rough estimates:

$$v = u \ln \frac{m_0}{m}.$$

The exhaust velocity is approximately equal to the specific impulse, so we can set:

u=3000 m/s.

The final mass of the satellite is m=50kg. The initial mass m_0 includes the mass m of the satellite itself and the fuel mass m_p:

$$m_0 = m + m_p.$$

Suppose that the satellite velocity v on the low orbit is equal to the first space velocity 7.91km/s =7910m/s.

We express the necessary fuel mass mp through the remaining parameters and calculate its value:

$$v = u \ln \frac{m + m_p}{m}, \quad \Rightarrow \frac{v}{u} = \ln\left(1 + \frac{m_p}{m}\right), \quad \Rightarrow 1 + \frac{m_p}{m} = e^{\frac{v}{u}}, \quad \Rightarrow m_p = m\left(e^{\frac{v}{u}} - 1\right)$$

$$= 50\left(e^{\frac{7910}{3000}} - 1\right) \approx 50\,(13.97 - 1) \approx 700\,[\mathrm{kg}]\,.$$

Thus, the fuel mass needed to launch the given satellite is 700kg (14 times more than the mass of the satellite itself). Of course, this is an estimation of the lower mass boundary, because it's based on the ideal rocket equation.

Problem 02-08

Estimate the acceleration of a rocket at the moment when the spacecraft reaches the orbit. Take the following parameters: the specific impulse (exhaust velocity), u=3000ms; the mass of the spacecraft on the orbit, m=5000kg; the fuel burn rate, μ=100kg/s

Solution

We again apply the ideal rocket equation:

$$v = u \ln \frac{m_0}{m}.$$

In this formula the rocket velocity v depends on the rocket mass m(t), which decreases as fuel is burning. We suppose for simplicity that the fuel burn rate is constant, so that the dependence of the mass on time is described by the linear function:

$$m\,(t) = m_0 - \mu t.$$

The final mass m of the spacecraft is known in the given problem. Assuming that the exhaust velocity is constant, we can write the ideal rocket equation in the form:

$$v(t) = u \ln \frac{m_0}{m_0 - \mu t}.$$

By differentiating with respect to t, we find the acceleration of the rocket:

$$\frac{dv}{dt} = a(t) = u \frac{1}{\frac{m_0}{m_0 - \mu t}} \cdot \frac{(-m_0)(-\mu)}{(m_0 - \mu t)^2} = \frac{u\,\cancel{(m_0 - \mu t)}\,\cancel{m_0}\,\mu}{\cancel{m_0}\,(m_0 - \mu t)^{\cancel{2}}} = \frac{u\mu}{m_0 - \mu t}.$$

Thus, we have obtained the rocket acceleration formula in the form:

$$a(t) = \frac{u\mu}{m_0 - \mu t}.$$

Notice that the acceleration a increases if any of the three parameters (the time t, the fuel burn rate μ and the exhaust velocity u) is increased. This can be easily proved by calculating the partial derivatives with respect to each of the variables. For example, the partial derivative in t is determined by the expression:

$$\frac{\partial a}{\partial t} = -\frac{u\mu}{(m_0 - \mu t)^2} \cdot (-\mu) = \frac{u\mu^2}{(m_0 - \mu t)^2} > 0.$$

Similarly, the partial derivative in μ is given by

$$\frac{\partial a}{\partial \mu} = \frac{u(m_0 - \mu t) - u\mu(-t)}{(m_0 - \mu t)^2} = \frac{u m_0 - \cancel{u\mu t} + \cancel{u\mu t}}{(m_0 - \mu t)^2} = \frac{u m_0}{(m_0 - \mu t)^2} > 0.$$

We can estimate the acceleration of the rocket when it reaches the orbit, assuming that the final mass is m=5000kg:

$$a = \frac{u\mu}{m_0 - \mu t} = \frac{u\mu}{m} = \frac{3000 \cdot 100}{5000} = 60\,\frac{\mathrm{m}}{\mathrm{s}^2} \approx 6\,g.$$

Thus, the rocket at the final stage of the trajectory can experience a big acceleration. Since it is a non-inertial frame of reference, the force of inertia of the same magnitude will affect the astronauts in the opposite direction (i.e. directed towards the Earth). As it can be seen, the positive g-forces, the astronauts can experience at take off can reach several g.

Self-Test 13

(1) Which of the following is NOT an example of motion of object with varying mass?

(A) an arrow shot from a bow

(B) a car

(C) movement of ice on ocean

(D) rocket

(2) The rocket equation is also known as

(A) Einstein formula

(B) Tsiolkovsky equation

(C) Chandrashekhar limit

(D) Kepler's law

(3) The rocket equation does not include

(A) exhaust velocity

(B) intial mass of the rocket

(C) maximum change in velocity

(D) acceleration due to gravity

Short Answer Questions 04

(1) Consider a person who is jumping by pushing his body against the floor. The pushing down of the floor gives his upwards push. Now consider an analogous case of a rocket which pushes a load of gas towards earth and gets upwards thrust. Similar downward push gives it upwards motion. However, in the empty space there is no floor to push. Will a spacecraft which is stationary in the empty space get motion by pushing jet of gases out of it?

(2) How much fuel (in kg) will be required to accelerate a rocket of mass 1 kg (without fuel) with a specific impulse (exhaust velocity) of 3000m/s to an escape velocity of 11 km/s as per Tsiolkovsky's formula.

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Course Companion Website

Visit Here for Course Companion website for this course:

UNIT 02-03: ROTATIONAL MOTION

LEARNING OBJECTIVES

After successful completion of this unit, you will be able to

- Explain the concepts of angular displacement, angular velocity and angular momentum
- Describe the concept of torque
- Elaborate the concept of conservation of angular momentum
- Explain the concept of conservation of angular momentum as applied to various every-day scenarios

INTRODUCTION

Fig 03-01: A formula one car driver makes a sharp curve. The angular motion is evident not only in the turn which the car is making on the road, but also in the rotation of wheels, steering wheel and in engine.

Many motions, such as the arc of a bird's flight or Earth's path around the Sun, are curved. Recall that Newton's first law tells us that motion is along a straight line at constant speed unless there is a net external force. We will therefore study not only motion along curves, but also the forces that cause it, which we call as 'torque'. In some ways, this Unit is a continuation of Unit 01-02 on Dynamics: Newton's Laws of Motion as we study more applications of Newton’s laws of motion.

I will introduce you to the concept of the basic angular motion. How we define angular displacement, angular velocity and angular momentum will be the starting point of the unit. Just as Newton’s law for linear motion says that change of linear momentum gives us Force, the change in angular momentum gives us external torque. For an isolated system which does not have external torque on it, the angular momentum is conserved. We will see a number of examples of manifestation of these principles in daily lives and in various devices like gyroscope.

03-01: ANGULAR DISPLACEMENT, ANGULAR VELOCITY AND ANGULAR MOMENTUM.

ANGULAR DISPLACEMENT

Angular displacement of a body is the angle (measured in either radians, degrees or revolutions) through which a point revolves around a centre or line, has been rotated in a specified ‘sense’ (like “clock-wise” or “anticlockwise”) about a specified axis. When an object rotates about its axis, the motion cannot simply be analyzed as a particle, as in circular motion it undergoes a change in velocity and acceleration at any time (t). When dealing with the rotation of an object, it becomes simpler to consider the body itself rigid. A body is generally considered rigid when the separations between all the particles remain constant throughout the objects motion. Thus, for example, parts of its mass are not flying off. In a realistic sense, all things can be deformable; however this impact is minimal and negligible. Thus the rotation of a rigid body over a fixed axis is referred to as rotational motion.

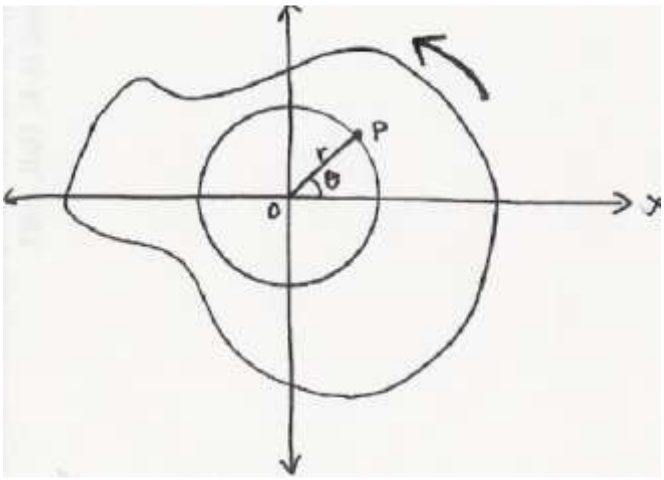


Fig 3-02: Rotation of a rigid object *P* about a fixed object about a fixed axis *O*

(Fig:https://en.wikipedia.org/wiki/Angular_displacement#/media/File:Angulardisplacement1.jpg)

In the example illustrated in adjacent figure, a particle or object P is at a fixed distance r from the origin, O, rotating counterclockwise. It becomes important to then represent the position of particle P in terms of its polar coordinates (r, θ). In this particular example, the value of θ is changing, while the value of the radius remains the same. (For the rectangular coordinates (x, y), both x and y vary with time). As the particle moves along the circle, it travels an arc length s, which becomes related to the angular position through the relationship:

$$s = r\theta$$

Angular displacement may be measured in radians or degrees. Using radians provides a very simple relationship between distance traveled around the circle and the distance r from the centre.

$$\theta = \frac{s}{r}$$

For example, if an object rotates 360° around a circle of radius r, the angular displacement is given by the distance traveled around the circumference - which is $2\pi r$ - divided by the radius: $\theta = 2\pi r/r$ which easily simplifies to: $\theta = 2\pi$. Therefore, 1 revolution is 2π radians.

When a particle travels from point P to point Q over δt, as it does in the illustration in the Figure, the radius of the circle goes through a change in angle $\Delta\theta = \theta_2 - \theta_1$ which equals the angular displacement.

ANGULAR VELOCITY

In physics, the angular velocity of a particle is the rate at which it rotates around a chosen center point: that is, the time rate of change of its angular displacement relative to the origin (i.e. in layman's terms: how quickly an object goes around something over a period of time - e.g. how fast the earth orbits the sun). It is measured in angle per unit time, radians per second in SI units, and is usually represented by the symbol omega (ω, sometimes Ω). By convention, positive angular velocity indicates counter-clockwise rotation, while negative is clockwise.

For example, a geostationary satellite completes one orbit per day above the equator, or 360 degrees per 24 hours, and has angular velocity $\omega = 360 / 24 = 15$ degrees per hour, or $2\pi / 24 \approx 0.26$ radians per hour. If angle is measured in radians, the linear velocity is the radius times the angular velocity, $v = r\omega$. With orbital radius 42,000 km from the earth's center, the satellite's speed through space is thus $v = 42{,}000 \times 0.26 \approx 11{,}000$ km/hr. The angular velocity is positive since the satellite travels eastward with the Earth's rotation (counter-clockwise from above the north pole.)

In three dimensions, angular velocity is a vector, with its magnitude measuring the rate of rotation, and its direction pointing along the axis of rotation (perpendicular to the radius and velocity vectors). The up-or-down orientation of angular velocity is conventionally specified by the right-hand rule.

Fig 03-03: Conventional direction of the axis of a rotating body

(Wikipedia:https://en.wikipedia.org/wiki/Right-hand_rule#/media/File:Right-hand_grip_rule.svg)

In mathematics a rotating body is commonly represented by a vector along the axis of rotation. The length of the vector gives the speed of rotation and the direction of the axis gives the direction of rotation according to the right-hand rule: right fingers curled in the direction of rotation and the right thumb pointing in the positive direction of the axis. This allows some easy calculations using the vector cross product. Note that no part of the body is moving in the direction of the axis arrow, which takes some getting used to. By coincidence, if your thumb points north the earth rotates according to the right-hand rule. This causes the sun and stars to appear to revolve according to the left-hand rule.

PARTICLE IN TWO DIMENSIONS

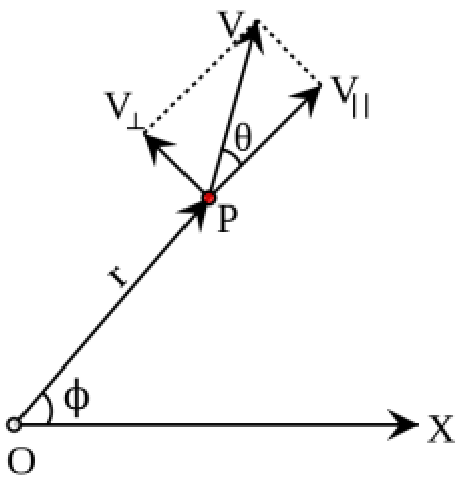


Fig 3-04: The angular velocity of the particle at P with respect to the origin O is determined by the perpendicular component of the velocity vector v.

(Source: Wikipedia, Angular Velocity, https://en.wikipedia.org/wiki/Angular_velocity#/media/File:Angular_velocity1.svg)

In the simplest case of circular motion at radius r, with position given by the angular displacement $\phi(t)$ from the x-axis, the angular velocity is the rate of change of angle with respect to time: $\omega = \phi(t)$. If ϕ is measured in radians, the distance from the x-axis around the circle to the particle is $\ell = r\phi$, and the linear velocity is $\mathbf{v}(t) = d\boldsymbol{\ell}/dt = \mathbf{r\omega}(t)$, so that $\omega = v/r$.

In the general case of a particle moving in the plane, the angular velocity is measured relative to a chosen center point, called the origin. The diagram shows the radius vector $\mathbf{r}$ from the origin O to the particle P, with its polar coordinates (r, ϕ). (All variables are functions of time t.) The particle has linear velocity splitting as $\mathbf{v} = \mathbf{v}_{\|} + \mathbf{v}_{\perp}$, with the radial component $\mathbf{v}_{\|}$ parallel to the radius, and the cross-radial (or tangential or circular) component $\mathbf{v}_{\perp}$ perpendicular to the radius. When there is no radial component, the particle moves around the origin in a circle. Similarly, when there is no cross-radial component, it moves in a straight line from the origin. Since radial motion leaves the angle unchanged, only the cross-radial component of linear velocity contributes to angular velocity.

The angular velocity ω is the rate of change of angle with respect to time, which can be computed from the cross-radial velocity as:

$$\omega = \frac{d\phi}{dt} = \frac{v_{\perp}}{r}.$$

Here the cross-radial speed $v_{\perp}$ is the signed magnitude of $\mathbf{v}_{\perp}$, positive for counter-clockwise motion, negative for clockwise. Taking polar coordinates for the linear velocity v gives magnitude v (linear speed) and angle θ relative to the radius vector; in these terms, $v_{\perp} = v\sin(\theta)$, so that

$$\omega = \frac{v\sin(\theta)}{r}.$$

These formulas may be derived from

$r = (x(t), y(t))$,

$v = (x'(t), y'(t))$ and

$\phi = \tan^{-1}(y(t) / x(t)) = \arctan(y(t) / x(t))$, [arctan means $\tan^{-1}$]

together with the projection formula

$$v_{\perp} = \frac{\mathbf{r}^{\perp}}{r} \cdot \mathbf{v},$$

Where $\mathbf{r}^{\perp} = (-y, x).$

In two dimensions, angular velocity is a number with plus or minus sign indicating orientation, but not pointing in a direction. The sign is conventionally taken to be positive if the radius vector turns counter-clockwise, and negative if clockwise. Angular velocity may be termed a 'pseudoscalar', a numerical quantity which changes sign under operations, such as inverting one axis or switching the two axes.

Angular Momentum

In physics, angular momentum (rarely, moment of momentum or rotational momentum) is the rotational equivalent of linear momentum. It is an important quantity in physics because it is a conserved quantity—the total angular momentum of a system remains constant unless acted on by an external torque.

In three dimensions, the angular momentum for a point particle is a quantity which is similar to a vector, hence it is also called a 'pseudo-vector'. Physical examples of pseudovectors include magnetic field, torque, angular velocity, and angular momentum. Let us not get into the discussion on 'pseudovectors' and how they differ from 'regular' vectors like linear velocity, etc. For the time being just remember that angular momentum is very much like a vector, but slightly different.

Angular momentum is given by $\mathbf{r} \times \mathbf{p}$, the cross product of the particle's position vector r (relative to some origin) and its momentum vector $\mathbf{p} = m\mathbf{v}$. This definition can be applied to each point in continuous materials like solids or fluids, or physical fields. Unlike momentum, angular momentum does depend on where the origin is chosen, since the particle's position is measured from it. The angular momentum vector of a point particle is parallel and directly proportional to the angular velocity vector ω of the particle (how fast its angular position changes), where the constant of proportionality depends on both the mass of the particle and its distance from origin.

Angular momentum is additive; the total angular momentum of a system is the (pseudo)vector sum of the angular momenta. For continua, i.e., bodies whose density is not constant but may vary from place to place or fields, one uses integration. The total angular momentum of any rigid body can be split into the sum of two main components: the angular momentum of the centre of mass (with a mass equal to the total mass) about the origin, plus the spin angular momentum of the object about the centre of mass.

Torque can be defined as the rate of change of angular momentum, analogous to force. The conservation of angular momentum helps explain many observed phenomena, for example the increase in rotational speed of a spinning figure skater as the skater's arms are contracted, the high rotational rates of neutron stars, the Coriolis effect, and precession of tops and gyroscopes. Applications include the gyrocompass, control moment gyroscope, inertial guidance systems, reaction wheels, flying discs or Frisbees, and Earth's rotation to name a few. In general, conservation does limit the possible motion of a system, but does not uniquely determine what the exact motion is.

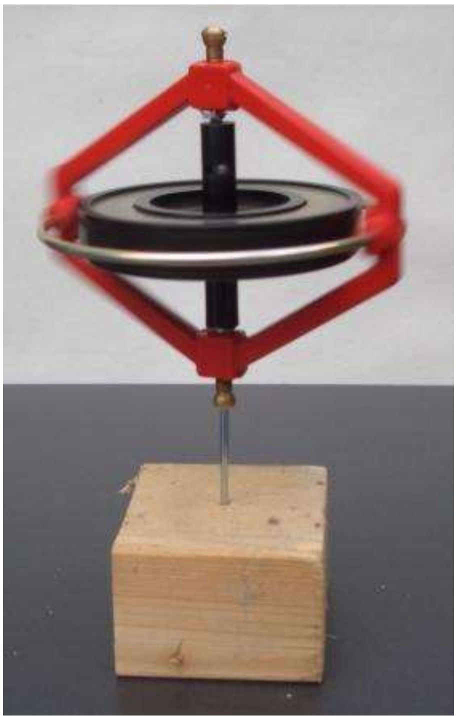
:

Fig3.05: Gyroskop

(https://en.wikipedia.org/wiki/Angular_momentum#/media/File:Gyroskop.jpg

SCALAR — ANGULAR MOMENTUM IN TWO DIMENSIONS

Velocity of the particle m with respect to the origin O can be resolved into components parallel to ($v_{||}$) and perpendicular to ($v_{\perp}$) the radius vector **r**. The angular momentum of m is proportional to the perpendicular component $v_{\perp}$ of the velocity, or equivalently, to the perpendicular distance $r\perp$ from the origin.

Angular momentum is a vector quantity (more precisely, a pseudovector) that represents the product of a body's rotational inertia (moment of inertia) and rotational velocity about a particular axis. However, if the particle's trajectory lies in a single plane, it is sufficient to discard the vector nature of angular momentum, and treat it as a scalar (more precisely, a pseudoscalar). Angular momentum can be considered a rotational analog of linear momentum. Thus, where linear momentum p is proportional to mass m m and linear speed v, p=mv, angular momentum **L** is proportional to moment of inertia I and angular speed ω.

$L = I\omega$.

Unlike mass, which depends only on amount of matter, moment of inertia is also dependent on the position of the axis of rotation and the shape of the matter. Unlike linear speed, which occurs in a straight line, angular speed occurs about a center of rotation. Therefore, strictly speaking, L should be referred to as the angular momentum relative to that center.

Because $I = r^2m$ for a single particle and $\omega = \frac{v}{r}$ for circular motion, angular momentum can be expanded, $L = r^2m.\frac{v}{r}$. This can be simplified to $L = r\ m\ v$, the product of the radius of rotation r and the linear momentum of the particle $p = m\ v$, where v in this case is the equivalent linear (tangential) speed at the radius ($= r\ \omega$).

This simple analysis can also apply to non-circular motion if only the component of the motion which is perpendicular to the radius vector is considered. In that case,

$L = r.m.v_\perp$, where $v_\perp = v\sin(\theta)$ is the perpendicular component of the motion. Expanding, $L = r\ m\ v\ \sin(\theta)$, rearranging, $L = r\ \sin(\theta)\ m\ v$, and reducing, angular momentum can also be expressed,

$L = r_\perp mv$, where $r_\perp = r\sin(\theta)$ is the length of the moment arm, a line dropped perpendicularly from the origin onto the path of the particle. It is this definition, (length of moment arm)×(linear momentum) to which the term moment of momentum refers.

SELF-TEST 14

(1) Angular displacement is a

(A) scalar quantity

(B) true vector

(C) pseudo-vector

(D) none of the above

(2) The direction of the angular velocity vector is

(A) in the direction of the linear velocity vector

(B) perpendicular to the plane of rotation given by right hand rule

(C) perpendicular to the plane of rotation given by left hand rule

(D) in the direction opposite to that of the linear velocity vector

(3) Rate of change of angular momentum gives

(A) angular velocity

(B) angular acceleration

(C) moment of inertia

(D) torque

SHORT ANSWER QUESTIONS 01

(1) Explain why angular displacement is not a true vector [search on internet to get and confirm your answer] [Hint See next question]

(2) Show that angular displacement in three-dimensions does not obey commutation law for large displacements.

(3) What is the angular velocity of the rotation of earth considering only the rotation around its axis (daily motion). What is its linear speed in m/s. [Obtain the data from internet, if required]

(4) What is the angular and linear speed of the rotation of earth considering only the yearly motion (rotation around the Sun). [Obtain the data from internet, if required]

(5) What is the importance of angular momentum?

03-02: TORQUE.

Torque is a measure of how much a force acting on an object causes that object to rotate. The object rotates about an axis, which we will call the pivot point, and will label 'O'. We will call the force '**F**'. The distance from the pivot point to the point where the force acts is called the moment arm, and is denoted by '**r**'. Note that this distance, '**r**', is also a vector, and points from the axis of rotation to the point where the force acts. (Refer to Figure for a pictorial representation of these definitions.).

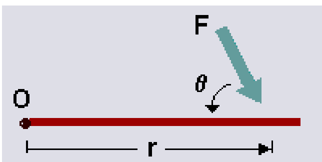


Fig 3-06: Definition of Torque

Torque is defined as

$\tau = \mathbf{r} \times \mathbf{F} = \mathbf{r}\,\mathbf{F} \sin(\theta)$.

In other words, torque is the cross product between the distance vector (the distance from the pivot point to the point where force is applied) and the force vector, 'a' being the angle between r and F.

Using the right hand rule, we can find the direction of the torque vector. If we put our fingers in the direction of r, and curl them to the direction of F, then the thumb points in the direction of the torque vector.

Imagine pushing a door to open it. The force of your push (F) causes the door to rotate about its hinges (the pivot point, O). How hard you need to push depends on the distance you are from the hinges (r) (and several other things, but let's ignore them now). The closer you are to the hinges (i.e. the smaller r is), the harder it is to push. This is what happens when you try to push open a door on the wrong side. The torque you created on the door is smaller than it would have been had you pushed the correct side (away from its hinges).

Note that the force applied, F, and the moment arm, r, are independent of the object. Furthermore, a force applied at the pivot point will cause no torque since the moment arm would be zero (r = 0).

Another way of expressing the above equation is that torque is the product of the magnitude of the force and the perpendicular distance from the force to the axis of rotation (i.e. the pivot point).

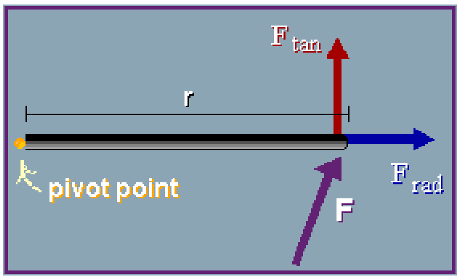


Fig 3-07: Radial and tangential component of the Force to rotate the door

Let the force acting on an object be broken up into its tangential (F_{tan}) and radial (F_{rad}) components (see Figure). (Note that the tangential component is perpendicular to the moment arm, while the radial component is parallel to the moment arm.) The radial component of the force has no contribution to the torque because it passes through the pivot point. So, it is only the tangential component of the force which affects torque (since it is perpendicular to the line between the point of action of the force and the pivot point).

There may be more than one force acting on an object, and each of these forces may act on different point on the object. Then, each force will cause a torque. The net torque is the sum of the individual torques.

Rotational Equilibrium is analogous to translational equilibrium, where the sum of forces is equal to zero. In rotational equilibrium, the sum of the torques is equal to zero. In other words, there is no net torque on the object.

Note that the SI unit of torque is newton-metre, which is also a way of expressing a Joule (the unit for energy). However, torque is not energy. So, to avoid confusion, we will use the units N.m, and not J. The distinction arises because energy is a scalar quantity, whereas torque is a vector.

Solved Problems

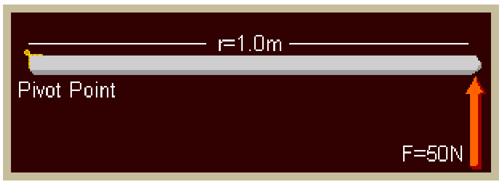


Fig 3-08: Problem illustration

In a hurry to catch a cab, you rush through a frictionless swinging door and onto the sidewalk. The force you exerted on the door was 50N, applied perpendicular to the plane of the door. The door is 1.0m wide. Assuming that you pushed the door at its edge, what was the torque on the swinging door (taking the hinge as the pivot point)?

Hints

1) Where is the pivot point?

2) What was the force applied?

3) How far from the pivot point was the force applied?

4) What was the angle between the door and the direction of force?

Solution

The pivot point is at the hinges of the door, opposite to where you were pushing the door. The force you used was 50N, at a distance 1.0 m from the pivot point. You hit the door perpendicular to its plane, so the angle between the door and the direction of force was 90 degrees. Since

$\tau = r \times F = r\,F \sin(\theta)$

the torque on the door was:

$\tau = (1.0m)\,(50N)\,\sin(90)$

$= 50$ N m

Note that this is only the magnitude of the torque; to complete the answer; we need to find the direction of torque. Using the right hand rule, we see that the direction of torque is out of the screen.

SELF-TEST 15

(1) In rotational motion the ….. is analogous to the force in translational motion

(A) torque

(B) moment of inertia

(C) angular momentum

(D) axis of rotation

(2) In rotational motion the ….. is analogous to the mass in translational motion

(A) torque

(B) moment of inertia

(C) angular momentum

(D) axis of rotation

03-03: CONSERVATION OF ANGULAR MOMENTUM.

The angular momentum of an isolated system remains constant in both magnitude and direction. The angular momentum is defined as the product of the moment of inertia I and the angular velocity. The angular momentum is a vector quantity and the vector sum of the angular momenta of the parts of an isolated system, is constant. This puts a strong constraint on the types of rotational motions which can occur in an isolated system. If one part of the system is given an angular momentum in a given direction, then some other part or parts of the system must simultaneously be given exactly the same angular momentum in the opposite direction. As far as we can tell, conservation of angular momentum is an absolute symmetry of nature. That is, we do not know of anything in nature that violates it.

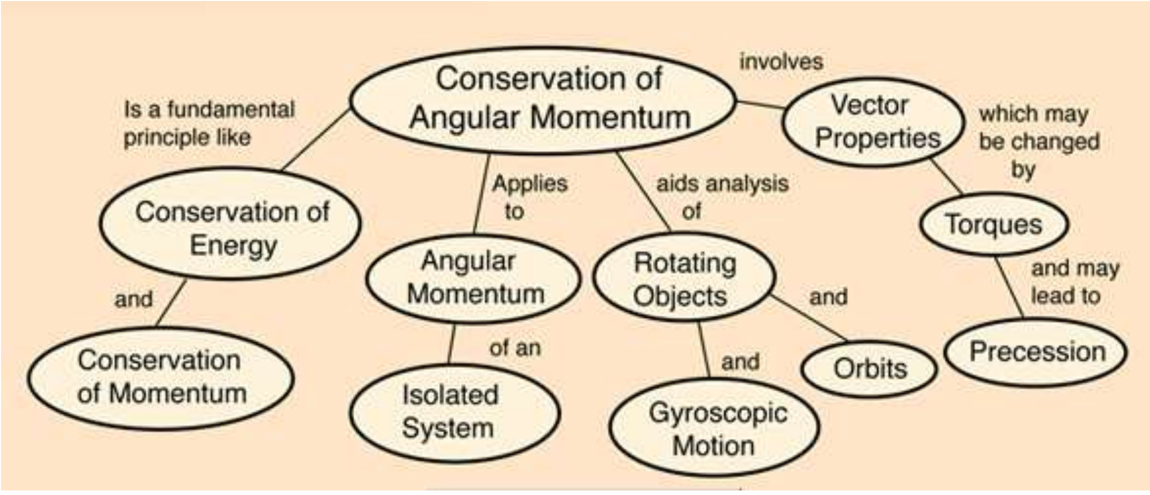
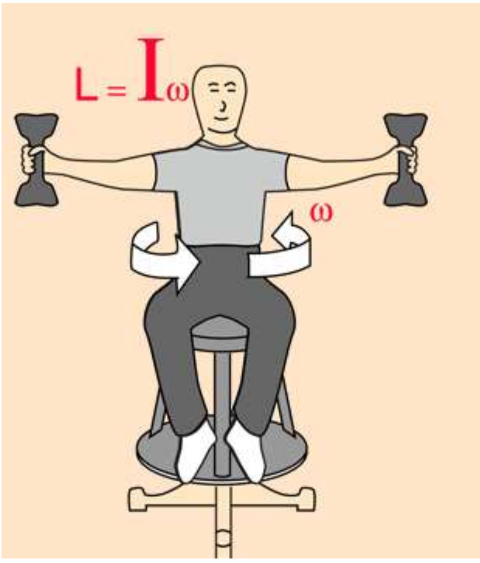


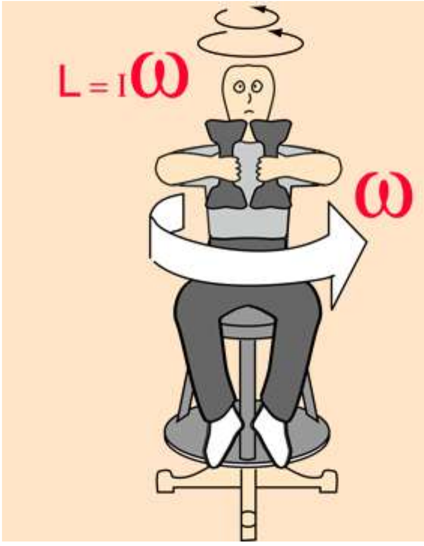
Fig 3-09: Various concepts in conservation of angular momentum

(**http://hyperphysics.phy-astr.gsu.edu/hbase/hframe.html**)

ROTATING CHAIR EXAMPLE



The moment of inertia is large with the masses held out. For a given angular momentum, the angular velocity is relatively low.



If the masses are pulled in, the moment of inertia is considerably decreased. Conservation of angular momentum dictates that the angular velocity must increase.

(**http://hyperphysics.phy-astr.gsu.edu/hbase/hframe.html**)

EXAMPLE 2: GYROSCOPE

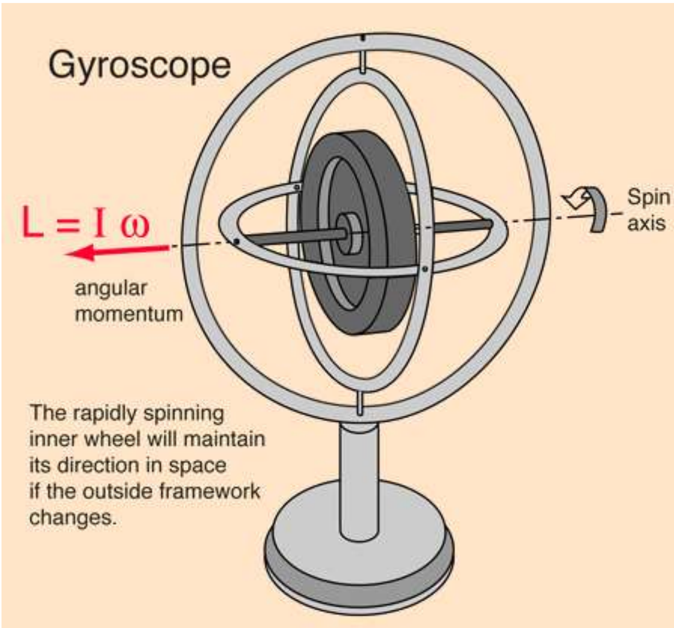


Fig3-10:Gyroscope is an example of conservation of angular momentum.

One typical type of gyroscope is made by suspending a relatively massive rotor inside three rings called gimbals. Mounting each of these rotors on high quality bearing surfaces insures that very little torque can be exerted on the inside rotor.

The classic image of a gyroscope is a fairly massive rotor suspended in light supporting rings called gimbals which have nearly frictionless bearings and which isolate the central rotor from outside torques. At high speeds, the gyroscope exhibits extraordinary stability of balance and maintains the direction of the high speed rotation axis of its central rotor. The implication of the conservation of angular momentum is that the angular momentum of the rotor maintains not only its magnitude, but also its direction in space in the absence of external torque. The classic type gyroscope finds application in gyro-compasses, but there are many more common examples of gyroscopic motion and stability. Spinning tops, the wheels of bicycles and motorcycles, the spin of the Earth in space, even the behavior of a boomerang are examples of gyroscopic motion.

Gyroscope Precession

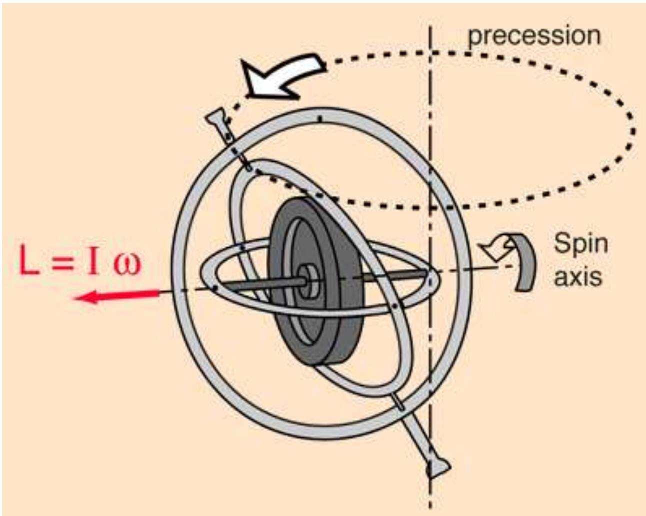


Fig 03-11: The precession of a gyroscope when it is disturbed

If a gyroscope is tipped, the gimbals will try to reorient to keep the spin axis of the rotor in the same direction. If released in this orientation, the gyroscope will precess in the direction shown because of the torque exerted by gravity on the gyroscope.

EXAMPLE OF PRECESSION OF SPINNING TOP

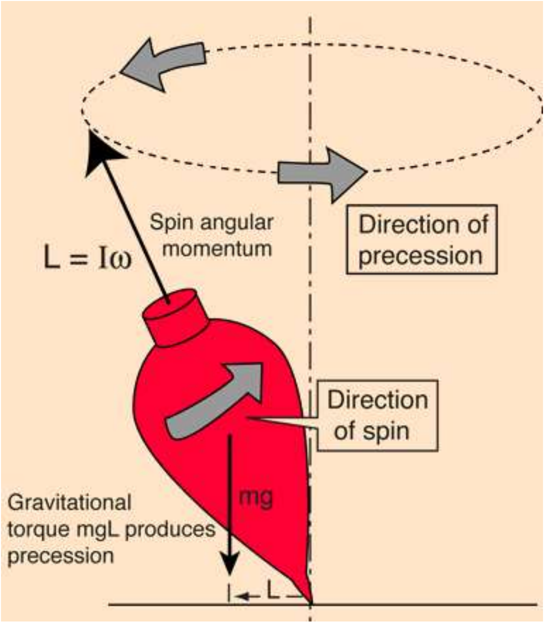


Fig 03-12: Gravity tries to trip the spinning top by exerting a force mg, producing a torque which has a direction perpendicular to the plane of the picture.

A rapidly spinning top will precess in a direction determined by the torque exerted by its weight. The precession angular velocity is inversely proportional to the spin angular velocity, so that the precession is faster and more pronounced as the top slows down.

The direction of the precession torque can be visualized with the help of the right-hand rule.Spin a top on a flat surface, and you will see its top end slowly revolve about the vertical direction, a process called precession. As the spin of the top slows, you will see this precession get faster and faster. It then begins to bob up and down as it precesses, and finally falls over. Showing that the precession speed gets faster, as the spin speed gets slower is a classic problem in mechanics. The process is summarized in the illustration below.

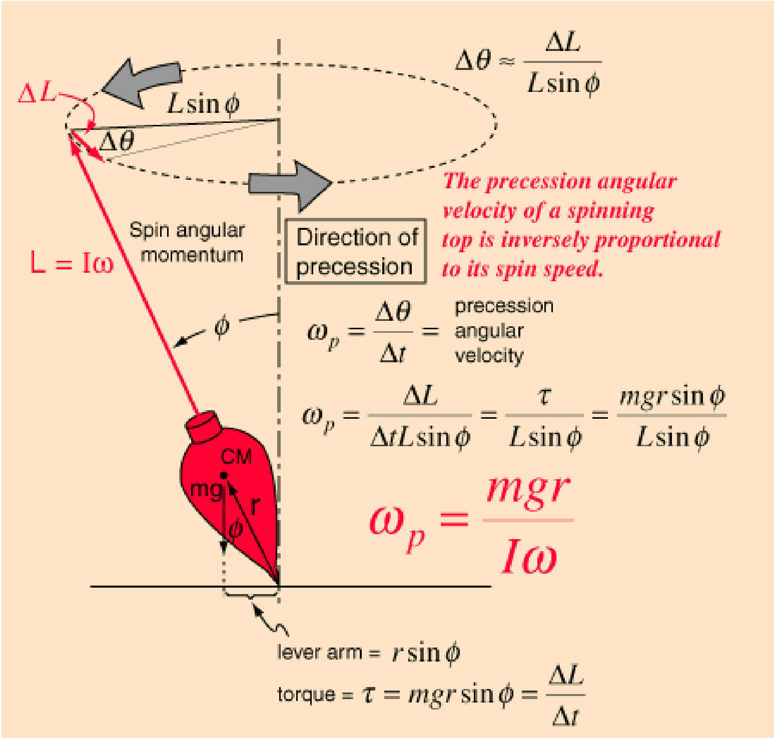


Fig 03-13: Precession of a spinning top

This process involves a considerable number of physical and mathematical concepts. The angular momentum of the spinning top is given by its moment of inertia times its spin speed but this exercise requires an understanding of it's vector nature. A torque is exerted about an axis through the top's supporting point by the weight of the top acting on its center of mass with a lever arm with respect to that support point. Since torque is equal to the rate of change of angular momentum, this gives a way to relate the torque to the precession process. From the definition of the angle of precession, the rate of change of the precession angle q can be expressed in terms of the rate of change of angular momentum and hence in terms of the torque.

The expression for precession angular velocity is valid only under the conditions where the spin angular velocity w is much greater than the precession angular velocity w_P. When the top slows down, the top begins to wobble, an indication that more complicated types of motion are coming into play.

EXAMPLE: BICYCLE WHEEL

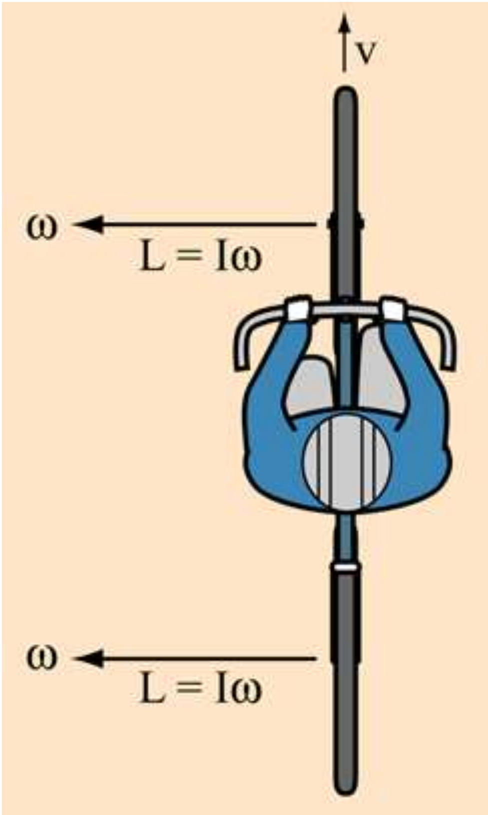


Fig 03-14: Top view of a bicycle rider

The angular momentum of the turning bicycle wheels makes them act like gyroscopes to help stabilize the bicycle. This gyroscopic action also helps to turn the bicycle.

Having pointed to the gyroscopic nature of the bicycle wheel, it should be pointed out that experiments indicate that the gyroscopic stability arising from the wheels is not a significant part of the stability of a bicycle. The moments of inertia and the speeds are not large enough. The experiments and review of Lowell and McKell indicate that the stability of the bicycle can be described in terms of centrifugal force. A rider who feels an unbalance to the left will turn the handlebars left, producing a segment of a circular path with resulting centrifugal force which pushes the top of the bicycle back toward vertical and a balanced condition.

Presumably the larger masses and speeds of motorcycle wheels do make the gyroscopic torques a much larger factor with motorcycles.

EXAMPLE PRECESSION OF SPINNING WHEEL

A spinning wheel which is held up by one end of its axle will precess in the sense shown if it has that direction of spin. If the spin is reversed, it will precess in the opposite direction. The sense of precession is determined by the direction of the torque due to the weight of the spinning wheel. That torque is perpendicular to the angular momentum of the wheel.

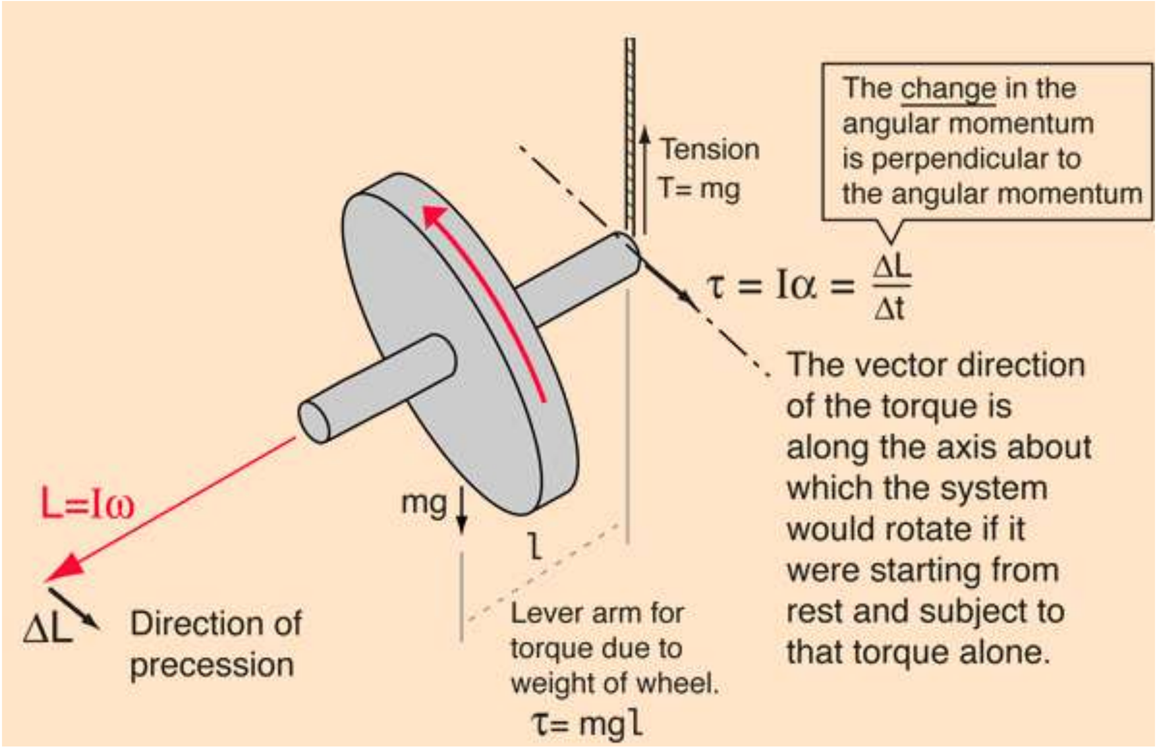


Fig 03-15: Example of spinning wheel

Self-Test 16

(1) The spinning top does not get toppled because

(A) torque given by earth's gravity (or the weight) is used in producing the precession

(B) angular momentum is conserved

(C) centripetal force balances the gravitational pull

(D) direction of angular momentum is perpendicular to the direction of the weight

(2) The axis of the spinning top executes a motion along the surface of a cone. This motion is called

(A) gyration

(B) precession

(C) ecliptic

(D) spin

(3) As the tops spins at slower rate, the angular speed of precession becomes

(A) large

(B) small

(C) infinite

(D) zero

Short Answer Questions 03

(1) Explain why the spinning top does not topple.

(2) Explain why a rotating wheel with very narrow width does not topple.

(3) Explain the principle of conservation of angular momentum. Give an example of non-conservation of angular momentum.

(4) Will the principle of conservation of angular momentum hold if the frames are linearly accelerating (like lift problem seen earlier)?

(5) A table is rotating with constant angular speed; a spinning top is placed on the table. Neglecting effect of gravity, friction, rotation of the earth and viscosity of air, find out its angular momentum in the frame of the table. Do the same exercise for the lab frame (in which the table is rotating). What do you conclude from these?

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https://www.edx.org/course/apr-physics-1-part-2-rotational-motion-ricex-advphy1-2x

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WIKIPEDIA

Angular Displacement

Angular Velocity

Angular Acceleration

Torque

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THEORY CREDIT 03

UNIT 03-01: GRAVITATION

LEARNING OBJECTIVES

After successful completion of this unit, you will be able to

- State and explain the Newton's law of Universal Gravitation
- Explain the concept of Central Force field
- Describe the concept of Uniform Circular Motion
- State the Kepler's laws of planetary motion
- Elaborate the concepts of geosynchronous orbits, geostationary orbits, weightlessness and
- Discuss the elementary ideas on Global Positioning System (GPS)

INTRODUCTION

Gravity is one of the most important forces on us. It helps us glued to the ground, helps us digest our foods, keeps things anchored on tables, cupboards etc. In the space away from Earth, there is negligible gravitational pull from the earth. There people get struck by various things like pens, cameras, etc which are not fastened to spacescrafts.

Gravity (from Latin gravitas, meaning 'weight'), or gravitation, is a natural phenomenon by which all things with mass or energy—including planets, stars, galaxies, and even light—are brought toward (or gravitate toward) one another. On Earth, gravity gives weight to physical objects, and the Moon's gravity causes the ocean tides. The gravitational attraction of the original gaseous matter present in the Universe caused it to begin coalescing, forming stars – and for the stars to group together into galaxies – so gravity is responsible for many of the large-scale structures in the Universe. Gravity has an infinite range, although its effects become increasingly weaker on farther objects.

We will be learning one of most beautiful theory proposed by Newton: the theory of Universal Gravitation. It is not that people did not know that earth was pulling objects before him. What is striking is that there is a simple equation which governs the gravity and the constant of gravity which relate masses to the force of gravity is universal. The nature of gravitational pull between the Earth and the Sun is same as it is between Sun and Jupiter or any other planet. In fact, all the objects in the Universe obey the same law of gravity.

Newton’s laws of motion require a body to move in a straight line with a uniform speed if there are no forces acting on it. However, the planets appear to be moving in orbits (as per Kepler, in elliptical orbits). Hence there must be some force which is causing the path to be ‘bent’. This force is the force of gravity.

The early work of gravity owes much to the concept of ‘heliocentric’ theory which was established by Galileo and the work of Kepler. Hence I will introduce concepts of Laws of Planetary motion without proof. I will also try to apply the principles learned in the unit to the motion of satellite in orbits. I will briefly introduce the various concepts like geosynchronous orbit, geostationary orbit and GPS.

This is a very interesting Unit and will concentrate on law of gravitation which is an example of inverse square law. Similar concept is applied to the Coulomb’s law of electrostatics and that of law for magnetic interactions. So brace yourself to learn these concepts and enjoy learning..

01-01: Newton’s Law of Gravitation. Motion of a particle in a central force field (motion is in a plane, angular momentum is conserved, areal velocity is constant).

Newton's law of universal gravitation

Newton's law of universal gravitation states that every particle attracts every other particle in the universe with a force which is directly proportional to the product of their masses and inversely proportional to the square of the distance between their centers. This is a general physical law derived from empirical observations by what Isaac Newton called inductive reasoning. It is a part of classical mechanics and was formulated in Newton's work Philosophiæ Naturalis Principia Mathematica ("the Principia"), first published on 5 July 1686. When Newton's book was presented in 1686 to the Royal Society, Robert Hooke made a claim that Newton had obtained the inverse square law from him.

In today's language, the law states that, every point mass attracts every other point mass by a force acting along the line intersecting both points. The force is proportional to the product of the two masses, and inversely proportional to the square of the distance between them.

The equation for universal gravitation thus takes the form:

$$F = G\frac{m_1 m_2}{r^2}$$

Here F is the gravitational force acting between two objects, m_1 and m_2 are the masses of the objects, r is the distance between the centers of their masses, and G is the gravitational constant.

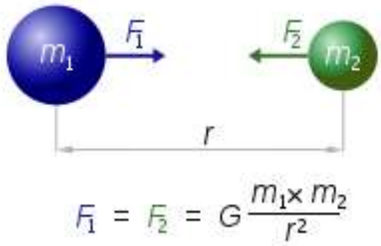

Fig 01-01: Newton’s law of Universal Gravitation

The first test of Newton's theory of gravitation between masses in the laboratory was the experiment conducted by the British scientist Henry Cavendish in 1798. It took place 111 years after the publication of Newton's Principia and approximately 71 years after his death.

Newton's law of gravitation resembles Coulomb's law of electrical forces, which is used to calculate the magnitude of the electrical force arising between two charged bodies. Both are inverse-square laws, where force is inversely proportional to the square of the distance between the bodies. Coulomb's law has the product of two charges in place of the product of the masses, and the electrostatic constant in place of the gravitational constant.

Newton's law has since been superseded by Albert Einstein's theory of general relativity, but it continues to be used as an excellent approximation of the effects of gravity in most applications. Relativity is required only when there is a need for extreme precision, or when dealing with very strong gravitational fields, such as those found near extremely massive and dense objects, or at very close distances (such as Mercury's orbit around the Sun)

Newton's law of universal gravitation can be written as a vector equation to account for the direction of the gravitational force as well as its magnitude. In this formula, quantities in bold represent vectors

$$\mathbf{F}_{21} = -G\frac{m_1 m_2}{|\mathbf{r}_{12}|^2}\,\hat{\mathbf{r}}_{12}$$

where

$\mathbf{F}_{21}$ is the force applied on object 2 exerted by object 1,

G is the gravitational constant,

m_1 and m_2 are respectively the masses of objects 1 and 2,

$|\mathbf{r}_{12}| = |\mathbf{r}_2 - \mathbf{r}_1|$ is the distance between objects 1 and 2, and

$\hat{\mathbf{r}}_{12} \stackrel{\text{def}}{=} \frac{\mathbf{r}_2 - \mathbf{r}_1}{|\mathbf{r}_2 - \mathbf{r}_1|}$ is the unit vector from object 1 to 2.

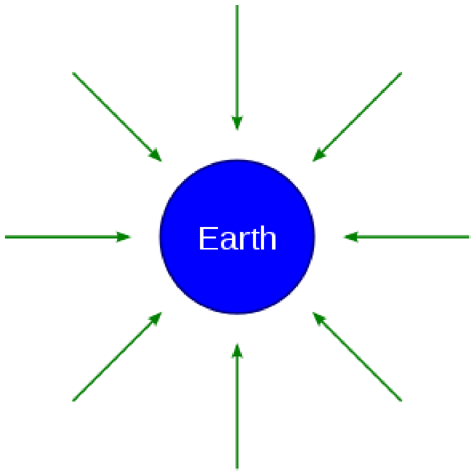

Fig 01-02: Gravity field surrounding Earth from a macroscopic perspective.

Motion of a Particle in a Central Force Field

In classical mechanics, the central-force problem is to determine the motion of a particle under the influence of a single central force. A central force is a force (possibly negative) that points from the particle directly towards a fixed point in space, the center, and whose magnitude only depends on the distance of the object to the center. In many important cases, the problem can be solved analytically, i.e., in terms of well-studied functions such as trigonometric functions.

The solution of this problem is important to classical physics, since many naturally occurring forces are central. Examples include gravity and electromagnetism as described by Newton's law of universal gravitation and Coulomb's law, respectively. The problem is also important because some more complicated problems in classical physics (such as the two-body problem with forces along the line connecting the two bodies) can be reduced to a central-force problem. Finally, the solution to the central-force problem often makes a good initial approximation of the true motion, as in calculating the motion of the planets in the Solar System.

The essence of the central-force problem is to solve for the position **r** of a particle moving under the influence of a central force F, either as a function of time t or as a function of the angle φ relative to the center of force and an arbitrary axis.

DEFINITION OF A CENTRAL FORCE

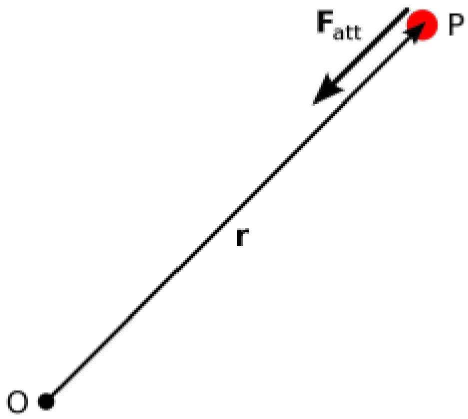

Fig 01-03: An attractive central force acting on a body at position P (shown in red). By definition, a central force must point either towards a fixed point O (if attractive) or away from it (if repulsive).

A conservative central force F has two defining properties. First, it must drive particles either directly towards or directly away from a fixed point in space, the center of force, which is often labeled O. In other words, a central force must act along the line joining O with the present position of the particle. Second, a conservative central force depends only on the distance r between O and the moving particle; it does not depend explicitly on time or other descriptors of position.

This two-fold definition may be expressed mathematically as follows. The center of force O can be chosen as the origin of a coordinate system. The vector r joining O to the present position of the particle is known as the position vector. Therefore, a central force must have the mathematical form

$$\mathbf{F} = F(r)\hat{\mathbf{r}}$$

where r is the vector magnitude |r| (the distance to the center of force) and $\hat{r}$ = **r**/|r| is the corresponding unit vector. According to Newton's second law of motion, the central force F generates a parallel acceleration a scaled by the mass m of the particle

$$\mathbf{F} = F(r)\hat{\mathbf{r}} = m\mathbf{a} = m\ddot{\mathbf{r}}$$

(Here two dots on the head of r denotes second derivative of r with respect to time i.e., acceleration.)

For attractive forces, F(r) is negative, because it works to reduce the distance r to the center. Conversely, for repulsive forces, F(r) is positive.

POTENTIAL ENERGY

A central force is always a conservative force; the magnitude $F(r)$ of a central force can always be expressed as the derivative of a time-independent potential energy function $U(r)$

$$F(r) = -\frac{dU}{dr}$$

Thus, the total energy of the particle—the sum of its kinetic energy and its potential energy U—is a constant; energy is said to be conserved. To show this, it suffices that the work W done by the force depends only on initial and final positions, not on the path taken between them.

$$W = \int_{\mathbf{r}_1}^{\mathbf{r}_2} \mathbf{F} \cdot d\mathbf{r} = \int_{\mathbf{r}_1}^{\mathbf{r}_2} F(r)\hat{\mathbf{r}} \cdot d\mathbf{r} = \int_{r_1}^{r_2} F dr = U(r_1) - U(r_2)$$

Since the scalar potential $V(r)$ depends only on the distance r to the origin, it has spherical symmetry.

ONE-DIMENSIONAL PROBLEM

If the initial velocity **v** of the particle is aligned with position vector **r**, then the motion remains forever on the line defined by **r**. This follows because the force—and by Newton's second law, also the acceleration **a**—is also aligned with **r**. To determine this motion, it is sufficient to solve the equation

$$m\frac{dr}{dt} = F(r)$$

One solution method is to use the conservation of total energy

$$E_{tot} = Kinetic\ Energy + Potential\ Energy$$

$$E_{tot} = \frac{1}{2}m\, v^2 + U(r)$$

$$v = |d\boldsymbol{r}/dx| = \sqrt{\frac{2}{m}(E_{tot} - U(r)}$$

Taking the reciprocal and integrating we get:

$$|t - t_0| = \sqrt{\frac{m}{2}} \int \frac{|dr|}{\sqrt{E_{\text{tot}} - U(r)}}$$

For the remainder of the article, it is assumed that the initial velocity **v** of the particle is not aligned with position vector **r**, so that the angular momentum vector **L** = **r** × *m* **v** is not zero.

Uniform circular motion

Every central force can produce uniform circular motion, provided that the initial radius **r** and speed **v** satisfy the equation for the centripetal force

$$\frac{mv^2}{r} = F(r)$$

If this equation is satisfied at the initial moments, it will be satisfied at all later times; the particle will continue to move in a circle of radius *r* at speed *v* forever.

Relation to the classical two-body problem

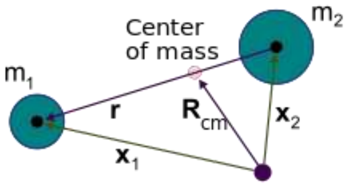

Fig 01-04: The positions $\mathbf{x}_1$ and $\mathbf{x}_2$ of two bodies can be expressed in terms of their relative separation **r** and the position of their center of mass $\mathbf{R}_{\text{cm}}$.

The central-force problem concerns an ideal situation (a"one-body problem") in which a single particle is attracted or repelled from an immovable point **O**, the center of force. However, physical forces are generally between two bodies; and by Newton's third law, if the first body applies a force on the second, the second body applies an equal and opposite force on the first. Therefore, both bodies are accelerated if a force is present between them; there is no perfectly immovable center of force. However, if one body is overwhelmingly more massive than the other, its acceleration relative to the other may be neglected; the center of the more massive body may be treated as approximately fixed. For example, the Sun is overwhelmingly more massive than the planet Mercury; hence, the Sun may be approximated as an immovable center of force, reducing the problem to the motion of Mercury in response to the force applied by the Sun. In reality, however, the Sun also moves (albeit only slightly) in response to the force applied by the planet Mercury.

Such approximations are unnecessary, however. Newton's laws of motion allow any classical two-body problem to be converted into a corresponding exact one-body problem. To demonstrate this, let $\mathbf{x}_1$ and $\mathbf{x}_2$ be the positions of the two particles, and let $\mathbf{r} = \mathbf{x}_1 - \mathbf{x}_2$ be their relative position. Then,

Relative acceleration $\frac{\mathrm{d}^2\mathrm{r}}{\mathrm{dt}^2} = \frac{\mathrm{d}^2\mathrm{x}_1}{\mathrm{dt}^2} - \frac{\mathrm{d}^2\mathrm{x}_2}{\mathrm{dt}^2}$

Using Newton's Second law:

$$\frac{d^2r}{dt^2} = \frac{d^2x_1}{dt^2} - \frac{d^2x_2}{dt^2} = \frac{F_{21}}{m_1} - \frac{F_{12}}{m_2}$$

Here F_{21} denotes Force on 2 exerted by 1. But Newton's third law tell us that, the force of the second body on the first body ($\mathbf{F}_{21}$) is equal and opposite to the force of the first body on the second ($\mathbf{F}_{12}$) i.e. $\mathbf{F}_{12} = -\mathbf{F}_{21}$. Hence

$$\frac{d^2r}{dt^2} = \left(\frac{1}{m_1} + \frac{1}{m_2}\right) F_{21}$$

If we define a quantity μ such that $\frac{1}{\mu} = \frac{1}{m_1} + \frac{1}{m_2}$,

$$\frac{d^2r}{dt^2} = \frac{F_{21}}{\mu}$$

Thus, the equation of motion for **r** can be written in the form

$$\mu\frac{d^2r}{dt^2} = F$$

This quantity μ is called the reduced mass of the system. It is the "effective" inertial mass appearing in the two-body problem of Newtonian mechanics. It is a quantity which allows the two-body problem to be solved as if it were a one-body problem. Reduced mass has the following properties:

The reduced mass is always less than or equal to the mass of each body:

$$\mu \leq m_1, \quad \mu \leq m_2$$

and has the reciprocal additive property:

$$\frac{1}{\mu} = \frac{1}{m_1} + \frac{1}{m_2}$$

which by re-arrangement is equivalent to half of the harmonic mean.

In the special case that $m_1 = m_2$:

$$\mu = \frac{m_1}{2} = \frac{m_2}{2}$$

If $m_1 \gg m_2$, then $\mu \approx m_2$.

As a special case, the problem of two bodies interacting by a central force can be reduced to a central-force problem of one body.

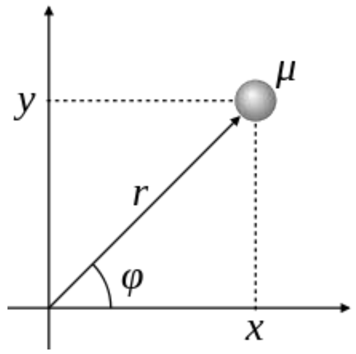

Fig 01-05: Any classical two-body problem to be converted into an equivalent one-body problem. The mass μ of the one equivalent body equals the reduced mass of the two original bodies, and its position **r** equals the difference of their positions.

Qualitative properties

Planar motion

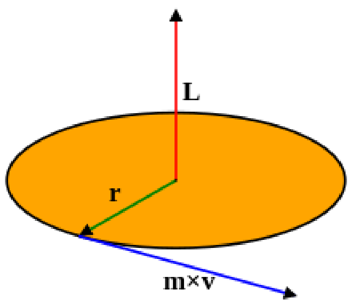

Fig 01-06: Illustration of planar motion. The angular momentum vector **L** is constant; therefore, the position vector **r** and velocity vector **v** must lie in the yellow plane perpendicular to **L**.

The motion of a particle under a central force **F** always remains in the plane defined by its initial position and velocity. This may be seen by symmetry. Since the position **r**, velocity **v** and force **F** all lie in the same plane, there is never an acceleration perpendicular to that plane, because that would break the symmetry between "above" the plane and "below" the plane.

To demonstrate this mathematically, it suffices to show that the angular momentum of the particle is constant. This angular momentum **L** is defined by the equation

$$\mathbf{L} = \mathbf{r} \times \mathbf{p} = \mathbf{r} \times m\mathbf{v}$$

where *m* is the mass of the particle and **p** is its linear momentum. Therefore, the angular momentum vector **L** is always perpendicular to the plane defined by the particle's position vector **r** and velocity vector **v**.

In general, the rate of change of the angular momentum **L** equals the net torque **r** × **F**

$$\frac{d\mathbf{L}}{dt} = \dot{\mathbf{r}} \times m\mathbf{v} + \mathbf{r} \times m\dot{\mathbf{v}} = \mathbf{v} \times m\mathbf{v} + \mathbf{r} \times \mathbf{F} = \mathbf{r} \times \mathbf{F} \,,$$

(Here as mentioned earlier, dot on the top of a variable denotes a derivative w.r.t. time t. Thus $\dot{r}$ denotes velocity and $\ddot{r}$ or $\dot{v}$ denotes acceleration.

The first term *m* **v** × **v** is always zero, because the vector cross product is always zero for any two vectors pointing in the same or opposite directions. However, when **F** is a central force, the remaining term **r** × **F** is also zero because the vectors **r** and **F** point in the same or opposite directions. Therefore, the angular momentum vector **L** is constant. Then

$$\mathbf{r} \cdot \mathbf{L} = \mathbf{r} \cdot (\mathbf{r} \times \mathbf{p}) = \mathbf{p} \cdot (\mathbf{r} \times \mathbf{r}) = 0$$

Consequently, the particle's position **r** (and hence velocity **v**) always lies in a plane perpendicular to **L**.

AREAL VELOCITY

In classical mechanics, areal velocity (also called sector velocity or sectorial velocity) is the rate at which area is swept out by a particle as it moves along a curve. In figure 01-07, suppose that a particle moves along the blue curve. At a certain time t, the particle is located at point B, and a short while later, at time t + Δt, the particle has moved to point C. The area swept out by the particle is the green area in the figure, bounded by the line segments AB and AC and the curve along which the particle moves. The areal velocity equals this area divided by the time interval Δt in the limit that Δt becomes vanishingly small. It is an example of a pseudovector (also called axial vector), pointing normal to the plane containing the position and velocity vectors of the particle.

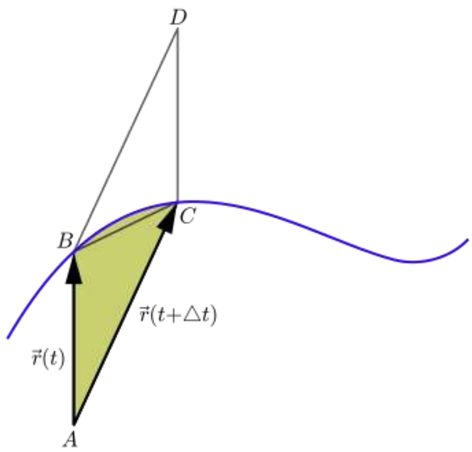

Fig 01-07: Areal velocity is the area swept out per unit time by a particle moving along a curve (shown in blue).

The concept of areal velocity is closely linked historically with the concept of angular momentum. Kepler's second law states that the areal velocity of a planet, with the sun taken as origin, is constant. Isaac Newton was the first scientist to recognize the dynamical significance of Kepler's second law. With the aid of his laws of motion, he proved in 1684 that any planet that is attracted to a fixed center sweeps out equal areas in equal intervals of time. By the middle of the 18th century, the principle of angular momentum was discovered gradually by Daniel Bernoulli and Leonhard Euler and Patrick d'Arcy. d'Arcy's version of the principle was phrased in terms of swept area. For this reason, the principle of angular momentum was often referred to in the older literature in mechanics as "the principle of equal areas." Since the concept of angular momentum includes more than just geometry, the designation "principle of equal areas" has been dropped in modern works.

Connection with the Angular Momentum

In the situation of the first figure 01-07, the area swept out during time period Δt by the particle is approximately equal to the area of triangle *ABC*. As Δt approaches zero this near-equality becomes exact as a limit.

Let the point *D* be the fourth corner of parallelogram *ABDC* shown in the figure 01-07 so that the vectors *AB* and *AC* add up by the parallelogram rule to vector *AD*. Then the area of triangle *ABC* is half the area of parallelogram *ABDC*, and the area of *ABDC* is equal to the magnitude of the cross product of vectors *AB* and *AC*. This area can also be viewed as a vector with this magnitude, pointing in a direction perpendicular to the parallelogram; this vector is the cross product itself:

$$\text{vector area of parallelogram } ABCD = \vec{r}(t) \times \vec{r}(t + \Delta t).$$

Hence

$$\text{vector area of triangle } ABC = \frac{\vec{r}(t) \times \vec{r}(t + \Delta t)}{2}.$$

The areal velocity is this vector area divided by Δt in the limit that Δt becomes vanishingly small:

$$\begin{aligned}\text{areal velocity} &= \lim_{\Delta t \to 0} \frac{\vec{r}(t) \times \vec{r}(t + \Delta t)}{2\Delta t} \\ &= \lim_{\Delta t \to 0} \frac{\vec{r}(t) \times \left(\vec{r}(t) + \vec{r}\,'(t)\Delta t\right)}{2\Delta t} \\ &= \lim_{\Delta t \to 0} \frac{\vec{r}(t) \times \vec{r}\,'(t)}{2} \left(\frac{\Delta t}{\Delta t}\right) \\ &= \frac{\vec{r}(t) \times \vec{r}\,'(t)}{2}.\end{aligned}$$

But, $\vec{r}\,'(t)$ is the velocity vector $\vec{v}(t)$ of the moving particle, so that

$$\frac{d\vec{A}}{dt} = \frac{\vec{r} \times \vec{v}}{2}.$$

On the other hand, the angular momentum of the particle is

$$\vec{L} = \vec{r} \times m\vec{v},$$

Conservation of areal velocity is a general property of **central force motion**, and, within the context of classical mechanics, is equivalent to the conservation of angular momentum.

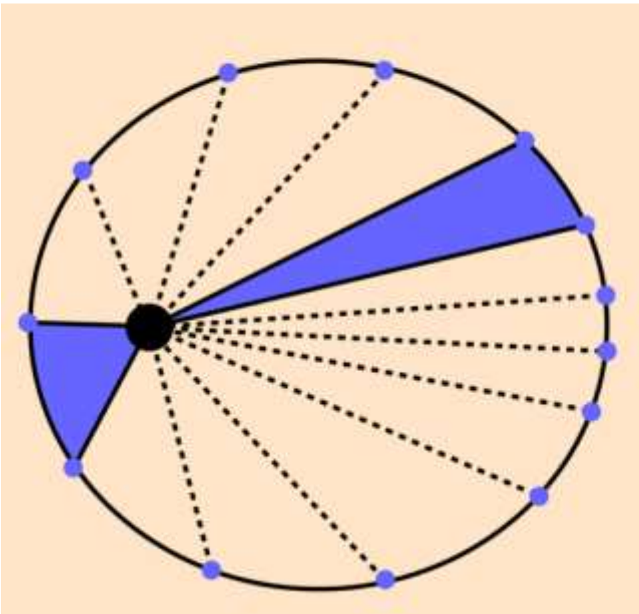

Fig 01-08: Illustration of Kepler's second law. The planet moves faster near the Sun, so the same area is swept out in a given time as at larger distances, where the planet moves more slowly.

Self-Test 17

(1) Central force is always

(A) conservative

(B) non-conservative

(C) uncertain

(D) infinite

(2) Planetary motion is an example of

(A) central force motion

(B) linear motion

(C) uniform motion

(D) unbound motion

(3) Keplar's second law is a consequence of

(A) inverse square law

(B) central force motion

(C) harmonic motion

(D) none of the above

Short Answer Questions 01

(1) What is meant by "universal" in the description "Newton's law of Universal Gravitation"?

(2) How could Newton conclude that all bodies in the universe attract each other? Did he do experiments on all the bodies in the universe?

(3) Compare the Newton's law of gravity with the Coulomb's law for electrostatic attraction. What are the similarities in the structure and nature of these laws? What are the differences?

(3) Compare attraction between two masses, two electrical charges and two magnets. What are the similarities and differences?

(4) The value of acceleration due to gravity gets modified in accelerated frames of reference. Does it violate the Newton's law of gravitation?

(5) What is meant by central force? Give at least five examples of central forces of varying sizes of systems [for example, nuclear, atomic, terrestrial (earth based), planetary and galactic sizes].

(6) What would have been the consequences if the mass had two signs (positive and negative) just like the electrical charges are positive and negative?

(7) What is the potential energy in a central force field.

(8) What is meant by areal velocity? How is it different from aerial velocity?

01-02: KEPLER'S LAWS (STATEMENT ONLY). SATELLITE IN CIRCULAR ORBIT AND APPLICATIONS.

In astronomy, Kepler's laws of planetary motion are three scientific laws describing the motion of planets around the Sun.

First Law: The orbit of a planet is an ellipse with the Sun at one of the two foci.

Second Law: A line segment joining a planet and the Sun sweeps out equal areas during equal intervals of time.

Third Law: The square of the orbital period of a planet is proportional to the cube of the semi-major axis of its orbit.

Let us see these laws using an illustration shown in the following figure.

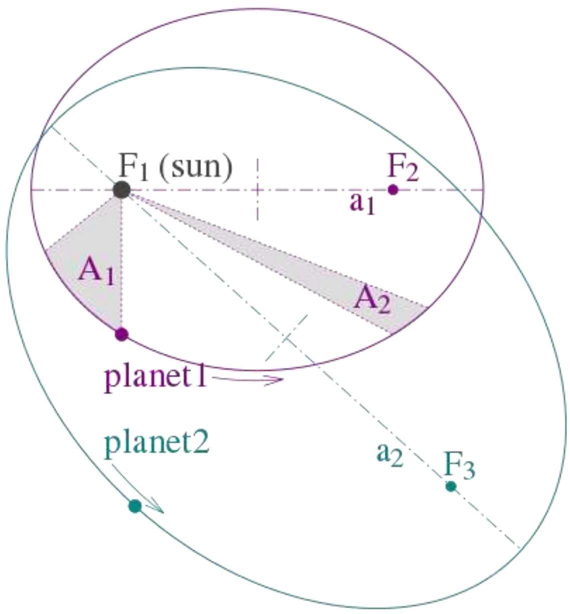

Fig 1.09: Illustration of Kepler's three laws with two planetary orbits.

(1) The orbits are ellipses, with focal points F_1 and F_2 for the first planet and F_1 and F_3 for the second planet. The Sun is placed in focal point F_1.

(2) The two shaded sectors A_1 and A_2 have the same surface area and the time for planet 1 to cover segment A_1 is equal to the time to cover segment A_2.

(3) The total orbit times for planet 1 and planet 2 have a ratio $(a_1/a_2)3/2$.

Most planetary orbits are nearly circular, and careful observation and calculation are required in order to establish that they are not perfectly circular. Calculations of the orbit of Mars, whose published values are somewhat suspect, indicate an elliptical orbit. From this, Johannes Kepler inferred that other bodies in the Solar System, including those farther away from the Sun, also have elliptical orbits.

Kepler's work (published between 1609 and 1619) improved the heliocentric theory of Nicolaus Copernicus, explaining how the planets' speeds varied, and using elliptical orbits rather than circular orbits with epicycles.

Isaac Newton showed in 1687 that relationships like Kepler's would apply in the Solar System to a good approximation, as a consequence of his own laws of motion and law of universal gravitation.

SATELLITE IN CIRCULAR ORBIT

Fig 01-10: International Space Station orbiting the earth is an artificial satellite as seen from Space Shuttle Endeavour

(https://en.wikipedia.org/wiki/Orbit#/media/File:STS-130_Endeavour_flyaround_5.jpg)

A satellite is any object that is orbiting the earth, sun or other massive body. Satellites can be categorized as natural satellites or man-made satellites. The moon, the planets and comets are examples of natural satellites. Accompanying the orbit of natural satellites are a host of satellites launched from earth for purposes of communication, scientific research, weather forecasting, intelligence, etc. Whether a moon, a planet, or some man-made satellite, every satellite's motion is governed by the same physics principles and described by the same mathematical equations.

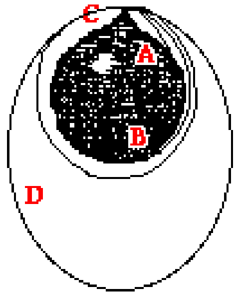

Fig 1-11: Satellite as a projectile which falls around the earth instead of into it

The fundamental principle to be understood concerning satellites is that a satellite is a projectile. That is to say, a satellite is an object upon which the only force is gravity. Once launched into orbit, the only force governing the motion of a satellite is the force of gravity. Newton was the first to theorize that a projectile launched with sufficient speed would actually orbit the earth. Consider a projectile launched horizontally from the top of the legendary Newton's Mountain - at a location high above the influence of air drag. As the projectile moves horizontally in a direction tangent to the earth, the force of gravity would pull it downward. If the launch speed was too small, it would eventually fall to earth. The diagram at the right resembles that found in Newton's original writings. Paths A and B illustrate the path of a projectile with insufficient launch speed for orbital motion. But if launched with sufficient speed, the projectile would fall towards the earth at the same rate that the earth curves. This would cause the projectile to stay the same height above the earth and to orbit in a circular path (such as path C). And at even greater launch speeds, a cannonball would once more orbit the earth, but now in an elliptical path (as in path D). At every point along its trajectory, a satellite is falling toward the earth. Yet because the earth curves, it never reaches the earth.

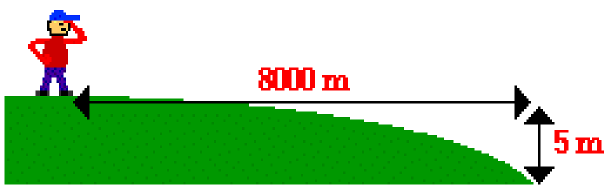

Fig 01-12: For every 8000m along the horizon the earth curves downward by 5 meters

So what launch speed does a satellite need in order to orbit the earth? The answer emerges from a basic fact about the curvature of the earth. For every 8000 meters measured along the horizon of the earth, the earth's surface curves downward by approximately 5 meters. So if you were to look out horizontally along the horizon of the Earth for 8000 meters, you would observe that the Earth curves downwards below this straight-line path a distance of 5 meters. For a projectile to orbit the earth, it must travel horizontally a distance of 8000 meters for every 5 meters of vertical fall. It so happens that the vertical distance that a horizontally launched projectile would fall in its first second is approximately 5 meters (0.5*g*t2). For this reason, a projectile launched horizontally with a speed of about 8000 m/s will be capable of orbiting the earth in a circular path. This assumes that it is launched above the surface of the earth and encounters negligible atmospheric drag. As the projectile travels tangentially a distance of 8000 meters in 1 second, it will drop approximately 5 meters towards the earth. Yet, the projectile will remain the same distance above the earth due to the fact that the earth curves at the same rate that the projectile falls. If shot with a speed greater than 8000 m/s, it would orbit the earth in an elliptical path.

VELOCITY, ACCELERATION AND FORCE VECTORS

The motion of an orbiting satellite can be described by the same motion characteristics as any object in circular motion. The velocity of the satellite would be directed tangent to the circle at every point along its path. The acceleration of the satellite would be directed towards the center of the circle - towards the central body that it is orbiting. And this acceleration is caused by a net force that is directed inwards in the same direction as the acceleration.

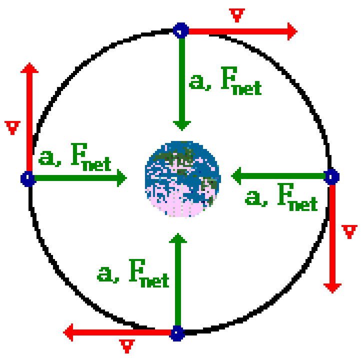

Fig 01-13: Acceleration, velocity and Force on a satellite.

This centripetal force is supplied by gravity - the force that universally acts at a distance between any two objects that have mass. Were it not for this force, the satellite in motion would continue in motion at the same speed and in the same direction. It would follow its inertial, straight-line path. Like any projectile, gravity alone influences the satellite's trajectory such that it always falls below its straight-line, inertial path. This is depicted in the fig 01-14. Observe that the inward net force pushes (or pulls) the satellite (denoted by blue circle) inwards relative to its straight-line path tangent to the circle. As a result, after the first interval of time, the satellite is positioned at position 1 rather than position 1'. In the next interval of time, the same satellite would travel tangent to the circle in the absence of gravity and be at position 2'; but because of the inward force the satellite has moved to position 2 instead. In the next interval of time, the same satellite has moved inward to position 3 instead of tangentially to position 3'. This same reasoning can be repeated to explain how the inward force causes the satellite to fall towards the earth without actually falling into it.

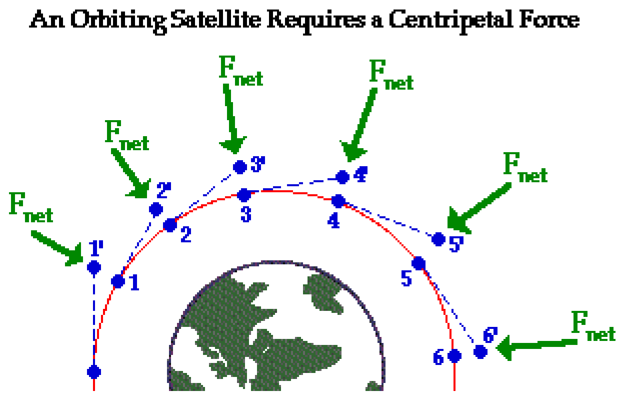

Fig 01-14: Role of centripetal force

ELLIPTICAL ORBITS OF SATELLITES

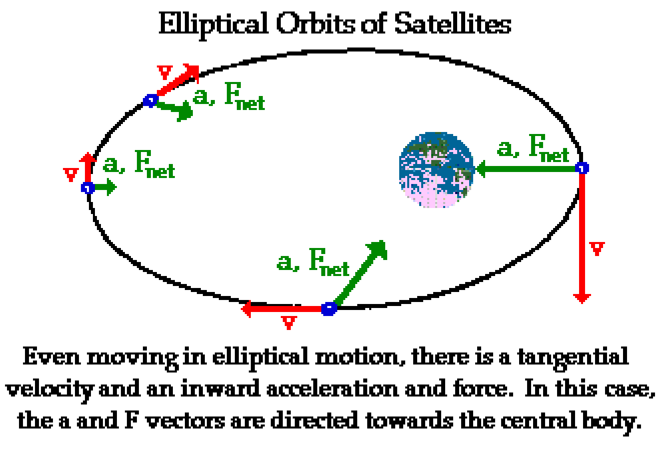

Fig 01-15: Elliptical Orbits

Occasionally satellites will orbit in paths that can be described as ellipses. In such cases, the central body is located at one of the foci of the ellipse. Similar motion characteristics apply for satellites moving in elliptical paths. The velocity of the satellite is directed tangent to the ellipse. The acceleration of the satellite is directed towards the focus of the ellipse. And in accordance with Newton's second law of motion, the net force acting upon the satellite is directed in the same direction as the acceleration - towards the focus of the ellipse. Once more, this net force is supplied by the force of gravitational attraction between the central body and the orbiting satellite. In the case of elliptical paths, there is a component of force in the same direction as (or opposite direction as) the motion of the object. Such a component of force can cause the satellite to either speed up or slow down in addition to changing directions. So unlike uniform circular motion, the elliptical motion of satellites is not characterized by a constant speed.

In summary, satellites are projectiles that orbit around a central massive body instead of falling into it. Being projectiles, they are acted upon by the force of gravity - a universal force that acts over even large distances between any two masses. The motion of satellites, like any projectile, is governed by Newton's laws of motion. For this reason, the mathematics of these satellites emerges from an application of Newton's universal law of gravitation to the mathematics of circular motion.

SELF-TEST 18

(1) The linear acceleration of a satellite always

(A) points perpendicular to the plane of rotation

(B) points away from the earth

(C) points towards the earth

(D) points in a straight line and never changes direction

(2) The centripetal force for the artificial satellite is supplied by

(A) fuel supply in the satellite

(B) initial thrust at the time of the launch

(C) solar cell

(D) gravity

(3) The speed required for a projectile to become a satellite is

(A) infinite

(B) called the escape velocity

(C) maximum if launched from the poles

(D) minimum if launched from the equator

SHORT ANSWER QUESTIONS 02

(1) State the three laws of planetary motion given by Kepler.

(2) Explain in details the first law of Kepler. Is it related with the Newton's laws of motion?

(3) Explain in details the second law given by Kepler.

(4) Explain in details the third law given by Kepler

01-03: Geosynchronous orbits, Weightlessness, Basic idea of global positioning system (GPS)

A geosynchronous satellite is a satellite in geosynchronous orbit, with an orbital period same as the Earth's rotation period. Such a satellite returns to the same position in the sky after each sidereal day, and over the course of a day traces out a path in the sky that is typically some form of analemma. A special case of geosynchronous satellite is the geostationary satellite, which has a geostationary orbit – a circular geosynchronous orbit directly above the Earth's equator. Another type of geosynchronous orbit used by satellites is the Tundra elliptical orbit.

Geosynchronous satellites have the advantage of remaining permanently in the same area of the sky, as viewed from a particular location on Earth, and so permanently within view of a given ground station. Geostationary satellites have the special property of remaining permanently fixed in exactly the same position in the sky, as viewed from any location on Earth, meaning that ground-based antennas do not need to track them but can remain fixed in one direction. Such satellites are often used for communication purposes; a geosynchronous network is a communication network based on communication with or through geosynchronous satellites.

Orbital characteristics

Circular Earth geosynchronous orbits have a radius of 42,164 km (26,199 mi). All Earth geosynchronous orbits, whether circular or elliptical, have the same semi-major axis. In fact, orbits with the same period always share the same semi-major axis:

$$a = \sqrt[3]{\mu\left(\frac{P}{2\pi}\right)^2}$$

where a is the semi-major axis, P is the orbital period, and μ is the geocentric gravitational constant, equal to 398,600.4418 km^3/s^2.

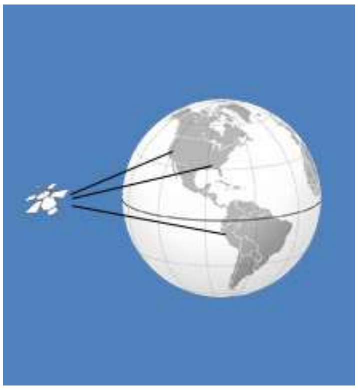

Fig 01-16: A geostationary satellite is in orbit around the Earth at an altitude where it orbits at the same rate as the Earth turns. An observer at any place where the satellite is visible will always see it in exactly the same spot in the sky, unlike stars and planets that move continuously.

(https://en.wikipedia.org/wiki/Geosynchronous_satellite#/media/File:Geostationary.png)

In the special case of a geostationary orbit, the ground track of a satellite is a single point on the equator. In the general case of a geosynchronous orbit with a non-zero inclination or eccentricity, the ground track is a more or less distorted figure-eight, returning to the same places once per sidereal day.

WEIGHTLESSNESS

Weightlessness, or an absence of weight, is an absence of stress and strain resulting from externally applied mechanical contact-forces, typically normal forces (from floors, seats, beds, scales, etc.). Counter intuitively, a uniform gravitational field does not by itself cause stress or strain, and a body in free fall in such an environment experiences no g-force acceleration and feels weightless. This is also termed zero-g, where the term is more correctly understood as meaning "zero g-force."

Fig 01-17: Astronauts at International Space Station experience zero g-force

(https://en.wikipedia.org/wiki/Weightlessness#/media/File:Foale_ZeroG.jpg)

When bodies are acted upon by non-gravitational forces- as in a centrifuge, a rotating space station, or within a space ship with rockets firing- a sensation of weight is produced, as the contact forces from the moving structure act to overcome the body's inertia. In such cases, a sensation of weight, in the sense of a state of stress can occur, even if the gravitational field were zero. In such cases, g-forces are felt, and bodies are not weightless.

When the gravitational field is non-uniform, a body in free fall suffers tidal effects and is not stress-free. Near a black hole, such tidal effects can be very strong. In the case of the Earth, the effects are minor, especially on objects of relatively small dimension (such as the human body or a spacecraft) and the overall sensation of weightlessness in these cases is preserved. This condition is known as microgravity and it prevails in orbiting spacecraft.

Fig 01-18: US astronaut Marsha Ivins demonstrates the effect of weightlessness on long hair

In October 2015, the NASA Office of Inspector General issued a health hazards report related to human spaceflight, including a human mission to Mars.

GLOBAL POSITIONING SATELLITE (GPS)

The Global Positioning System (GPS), originally Navstar GPS, is a satellite-based radio navigation system owned by the United States government and operated by the United States Air Force. It is a global navigation satellite system that provides geo location and time information to a GPS receiver anywhere on or near the Earth where there is an unobstructed line of sight to four or more GPS satellites. Obstacles such as mountains and buildings block the relatively weak GPS signals.

The GPS does not require the user to transmit any data, and it operates independently of any telephonic or internet reception, though these technologies can enhance the usefulness of the GPS positioning information. The GPS provides critical positioning capabilities to military, civil, and commercial users around the world. The United States government created the system, maintains it, and makes it freely accessible to anyone with a GPS receiver.

The GPS project was launched by the U.S. Department of Defense in 1973 for use by the United States military and became fully operational in 1995. It was allowed for civilian use in the 1980s. Advances in technology and new demands on the existing system have now led to efforts to modernize the GPS and implement the next generation of GPS Block IIIA satellites and Next Generation Operational Control System (OCX). Announcements from Vice President Al Gore and the White House in 1998 initiated these changes. In 2000, the U.S. Congress authorized the modernization effort, GPS III. During the 1990s, GPS quality was degraded by the United States government in a program called "Selective Availability"; this was discontinued in May 2000 by a law signed by President Bill Clinton. New GPS receiver devices using L5 frequency to begin release in 2018 are expected to have a much higher accuracy and pinpoint to a device within 30 centimeters or just less than one foot.

The GPS system is provided by the United States government, which can selectively deny access to the system, as happened to the Indian military in 1999 during the Kargil War, or degrade the service at any time. As a result, several countries have developed or are in the process of setting up other global or regional navigation systems. The Russian Global Navigation Satellite System (GLONASS) was developed contemporaneously with GPS, but suffered from incomplete coverage of the globe until the mid-2000s. GLONASS can be added to GPS devices, making more satellites available and enabling positions to be fixed more quickly and accurately, to within two meters. China's BeiDou Navigation Satellite System is due to achieve global reach in 2020. There are also the European Union Galileo positioning system, and India's NAVIC. Japan's Quasi-Zenith Satellite System (scheduled to commence in November 2018) will be a GPS satellite-based augmentation system to enhance GPS's accuracy.

Fig 01-19: Civilian GPS receivers ("GPS navigation device") in a marine application.

How GPS works

In June of 1993 the last of the 24 satellites of the Global Positioning System was placed into orbit, completing a satellite network capable providing position data to locate you anywhere on Earth within 30 meters. The satellites carry up to four cesium and rubidium atomic clocks which are periodically updated from a ground station in Colorado. The satellites transmit timing signals and position data. A GPS receiver, which can be a small, hand-held device, decodes the timing signals from several of the satellites, interpreting the arrival times in terms of latitude, longitude, and altitute with an uncertainty which may be as small as 10 meters. In differential mode, accuracies of less than a centimeter can be obtained for distances of hundreds of kilometers. Hand-held units read out longitude and latitude to a thousandth of an arc minute and change in the last decimal place in a couple of paces while walking. The satellites are in orbits much lower than the syncom satellites, orbiting around 17.7 million meters (11000 mi) above the earth, with orbit periods of the order of 10 hours.

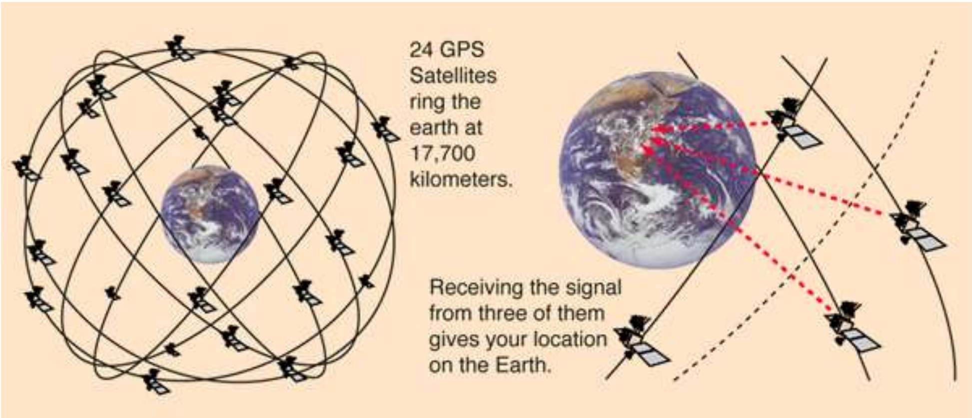

Fig 01-20: Working of GPS

From aircraft and ship navigation to finding the favorite fishing hole on the lake, the applications of GPS have multiplied rapidly.

Self-Test 19

(1) GPS works using the principles of

(A) triangulation

(B) radioactive decay

(C) cellular telephony

(D) Kepler's law

(2) At least …. Satellites are required for determination of location using GPS

(A) twenty four

(B) one

(C) two

(D) three

Short Answer Questions 03

(1) What is the difference between Geosynchronous and geostationary orbits.

(2) What is the importance of geostationary orbits.

(3) What is the concept of weightlessness? Can we also experience ‘masslessness’ under some circumstances?

(4) What is the importance of GPS and how does it work?

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YouTube Videos

Gravity Introduced https://www.youtube.com/watch?v=Xcel427Ezl0

Kepler’s Laws https://www.youtube.com/watch?v=kyR6EO_RMKE

How GPS works https://www.youtube.com/watch?v=FU_pY2sTwTA

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Classical central-force problem

Areal velocity

Geosynchronous orbit

Geosynchronous satellite

OER

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Course Companion Website

Visit Here for Course Companion website for this course:

UNIT 03-02: OSCILLATIONS

LEARNING OBJECTIVES

After successful completion of this unit, you will be able to

- Explain the concept of Simple Harmonic Motion
- Write the differential equation for SHM
- Discuss the various examples of SHM
- Write the expressions for kinetic and potential energies associated with SHM
- Discuss the phenomenon of damped oscillations
- Explain the definition of Q factor in relation to the oscillations in resonators

INTRODUCTION

Fig 02.01: There are at least four types of waves in this picture—only the water waves are evident. There are also sound waves, light waves, and waves on the guitar strings. The study of Simple Harmonic Motion forms the foundation for study of waves (credit: John Norton)

To instill some interest in the topic which you will be studying along with me, let me ask you a thought-provoking question. “What would have happened if there were no repetition in the nature?” To appreciate my question, you may list down all the phenomena which you observe which have a repetitive nature.

You may have listed cycles of seasons (four seasons taking turn in a pre-decided manner), day-and-night, phases of moons, birth-and-death, monthly cycle of getting salary on first of the month and having empty pocket on last day of the month, and so on.

What would have happened if there were no such repetition in the nature? Everything would have happened only once and would not have happened again. I doubt if there would be phenomena called life (birth-and-death) and hence I would not have been there to write this book and you would not be there to read it. What is more, there would not be light (electromagnetic oscillations) or gravity (no gravitons!) and no nature the way we know it.

So, just as 'change' is one of the facets of nature, so is repetition in time and space. Crystals are example of the repetition in space and periodic phenomena are example of repetitions in time.

Well you know that some repetitive processes have fixed period of repetition. Waves and oscillations are examples of such phenomena. One of the simplest model of waves is called simple harmonic oscillation. We will be studying this interesting phenomenon in the present Unit.

The concepts which you will learn in this Unit will be very useful in your development as a professional in physics. The concepts of oscillations are fundamental in understanding of various phenomena in physics including quantum mechanics, and electromagnetic theory.

02-01: SIMPLE HARMONIC MOTION, DIFFERENTIAL EQUATION OF SHM AND ITS SOLUTIONS.

In mechanics and physics, simple harmonic motion is a special type of periodic motion or oscillation motion where the restoring force is directly proportional to the displacement and acts in the direction opposite to that of displacement.

Simple harmonic motion can serve as a mathematical model for a variety of motions, such as the oscillation of a spring. In addition, other phenomena can be approximated by simple harmonic motion, including the motion of a simple pendulum as well as molecular vibration. Simple harmonic motion is typified by the motion of a mass on a spring when it is subject to the linear elastic restoring force given by Hooke's Law. The motion is sinusoidal in time and demonstrates a single resonant frequency. For simple harmonic motion to be an accurate model for a pendulum, the net force on the object at the end of the pendulum must be proportional to the displacement. This is a good approximation when the angle of the swing is small.

Simple harmonic motion provides a basis for the characterization of more complicated motions through the techniques of Fourier analysis. Therefore it can be simply defined as the periodic motion of a body along a straight line, such that the acceleration is directed towards the center of the motion and also proportional to the displacement from that point.

The motion of a particle moving along a straight line with an acceleration whose direction is always towards a fixed point on the line and whose magnitude is proportional to the distance from the fixed point is called simple harmonic motion [SHM].

Simple harmonic motion is shown in real space in Fig 02-02. The orbit is periodic. (Here the velocity and position axes have been reversed from the standard convention to align the two diagrams)

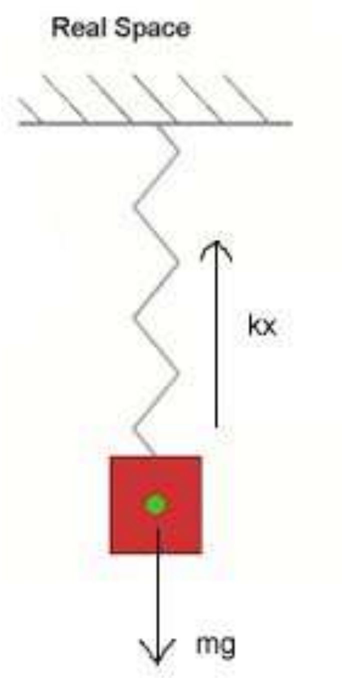

Fig 02-02: A spring opposes the motion by a restoring force (kx) in direction opposing weight mg

In the diagram, a simple harmonic oscillator, consisting of a weight attached to one end of a spring, is shown. The other end of the spring is connected to a rigid support such as a wall. If the system is left at rest at the equilibrium position then there is no net force acting on the mass. However, if the mass is displaced from the equilibrium position, the spring exerts a restoring elastic force that obeys Hooke's law.

Mathematically, the restoring force F is given by

$F = -kx$,

where F is the restoring elastic force exerted by the spring (in SI units: N), k is the spring constant ($N \cdot m^{-1}$), and x is the displacement from the equilibrium position.

For any simple mechanical harmonic oscillator:

When the system is displaced from its equilibrium position, a restoring force that obeys Hooke's law tends to restore the system to equilibrium.

Once the mass is displaced from its equilibrium position, it experiences a net restoring force. As a result, it accelerates and starts going back to the equilibrium position. When the mass moves closer to the equilibrium position, restoring force decreases. At the equilibrium position, the net restoring force vanishes. However, at x = 0, the mass has momentum because of the acceleration that the restoring force has imparted. Therefore, the mass continues past the equilibrium position, compressing the spring. A net restoring force then slows it down until its velocity reaches zero, whereupon it is accelerated back to the equilibrium position again.

As long as the system has no energy loss, the mass continues to oscillate. Thus simple harmonic motion is a type of periodic motion.

In Newtonian mechanics, for one-dimensional simple harmonic motion, the equation of motion, which is a second-order linear ordinary differential equation with constant coefficients, can be obtained by means of Newton's 2nd law and Hooke's law for a mass on a spring.

$$F_{\text{net}} = m\frac{\mathrm{d}^2 x}{\mathrm{d}t^2} = -kx,$$

where m is the inertial mass of the oscillating body, x is its displacement from the equilibrium (or mean) position, and k is a constant (the spring constant for a mass on a spring).

Therefore,

$$\frac{\mathrm{d}^2 x}{\mathrm{d}t^2} = -\frac{k}{m}x,$$

Solving the differential equation above produces a solution that is a sinusoidal function.

$$x(t) = c_1 \cos(\omega t) + c_2 \sin(\omega t)$$

This can be written in form

$$x(t) = A\cos(\omega t - \varphi),$$

where

$$\omega = \sqrt{\frac{k}{m}}, \qquad A = \sqrt{{c_1}^2 + {c_2}^2}, \qquad \tan\varphi = \frac{c_2}{c_1},$$

In the solution, c_1 and c_2 are two constants determined by the initial conditions, and the origin is set to be the equilibrium position.[A] Each of these constants carries a physical meaning of the motion: A is the amplitude (maximum displacement from the equilibrium position), $\omega = 2\pi f$ is the angular frequency, and φ is the phase.[B]

Using the techniques of calculus, the velocity and acceleration as a function of time can be found:

$$v(t) = \frac{\mathrm{d}x}{\mathrm{d}t} = -A\omega\sin(\omega t - \varphi),$$

Speed:

$$\omega\sqrt{A^2 - x^2}$$

Maximum speed: $v = \omega A$ (at equilibrium point)

$$a(t) = \frac{\mathrm{d}^2x}{\mathrm{d}t^2} = -A\omega^2\cos(\omega t - \varphi).$$

Maximum acceleration: $A\omega^2$ (at extreme points)

By definition, if a mass m is under SHM its acceleration is directly proportional to displacement.

$$a(x) = -\omega^2 x.$$

where

$$\omega^2 = \frac{k}{m}$$

Since $\omega = 2\pi f$,

$$f = \frac{1}{2\pi}\sqrt{\frac{k}{m}},$$

and, since $T = \frac{1}{f}$ where T is the time period,

$$T = 2\pi\sqrt{\frac{m}{k}}.$$

These equations demonstrate that the simple harmonic motion is isochronous (the period and frequency are independent of the amplitude and the initial phase of the motion).

SELF-TEST 20

(1) The differential equation of the SHM is of the form

(A) $\frac{d^2y}{dx^2} + ky = 0$

(B) $\frac{d^2y}{dx^2} - ky = 0$

(C) $\frac{d\,y}{dx} + ky = 0$

(D) $\frac{d\,y}{dx} - ky = 0$

(2) During the SHM, the acceleration always

(A) points to the normal to the place of motion as per Left hand rule

(B) points to the normal to the plane of motion as per rigtht hand rule

(C) points towards the mean point of motion

(D) points away from the mean point of motion

(3) The angular velocity for SHM given by differential equation $\frac{d^2y}{dx^2} + y = 0$ is

(A) unity

(B) zero

(C) infinite

(D) 2 π

SHORT ANSWER QUESTIONS 01

(1) What are the conditions for Simple Harmonic Motion?

(2) Do the forces acting in cases of SHM necessarily satisfy the conditions of a central force?

(3) Is the motion under SHM always periodic? Can there be periodic motion which are not simple harmonic?

02-02: Kinetic and Potential Energy, Total Energy and their time averages.

We can find the kinetic energy and potential energy of a system performing simple harmonic motion.

We will assume as in the previous example the example of a spring attached to a mass m as an example of simple harmonic oscillation. The kinetic energy is given as ½ mv^2 and potential energy is given by ½ kx^2.

Substituting ω^2 with k/m, the kinetic energy K of the system at time t is

$$K(t) = \tfrac{1}{2}mv^2(t) = \tfrac{1}{2}m\omega^2 A^2 \sin^2(\omega t + \varphi) = \tfrac{1}{2}kA^2 \sin^2(\omega t + \varphi),$$

and the potential energy is

$$U(t) = \tfrac{1}{2}kx^2(t) = \tfrac{1}{2}kA^2 \cos^2(\omega t + \varphi).$$

In the absence of friction and other energy loss, the total mechanical energy has a constant value

$$E = K + U = \tfrac{1}{2}kA^2.$$

Examples of SHM

The following physical systems are some examples of simple harmonic oscillator.

Mass on a spring

Fig 02-03: Mass suspended on spring

A mass m attached to a spring of spring constant k exhibits simple harmonic motion in closed space. The equation for describing the period

$$T = 2\pi\sqrt{\frac{m}{k}}$$

is independent of both the amplitude and gravitational acceleration. The above equation is also valid in the case when an additional constant force is being applied on the mass, i.e. the additional constant force cannot change the period of oscillation.

UNIFORM CIRCULAR MOTION

Simple harmonic motion can be considered the one-dimensional projection of uniform circular motion. If an object moves with angular speed ω around a circle of radius r centered at the origin of the xy-plane, then its motion along each coordinate is simple harmonic motion with amplitude r and angular frequency ω.

MASS OF A SIMPLE PENDULUM

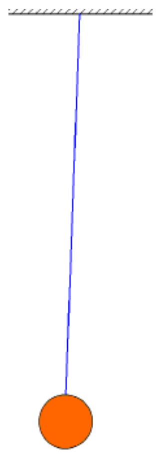

Fig 02-04: A Pendulum performs simple harmonic oscillations in small angle approximation

The motion of an undamped pendulum approximates to simple harmonic motion if the angle of oscillation is small. In the small-angle approximation, the motion of a simple pendulum is approximated by simple harmonic motion. When the motion of simple pendulum with larger angle is considered, the motion is no more simple harmonic. This is because the approximation that sine of angle is equal to angle is not valid for large value of angle. The period of a mass attached to a pendulum of length l with gravitational acceleration g is given by

$$T = 2\pi\sqrt{\frac{l}{g}}$$

This shows that the period of oscillation is independent of the amplitude and mass of the pendulum but not of the acceleration due to gravity, g, therefore a pendulum of the same length on the Moon would swing more slowly due to the Moon's lower gravitational field strength. Because the value of g varies slightly over the surface of the earth, the time period will vary slightly from place to place and will also vary with height above sea level.

This approximation is accurate only for small angles because of the expression for angular acceleration α being proportional to the sine of the displacement angle:

$$-m g l \sin\theta = I\alpha ,$$

where I is the moment of inertia. When θ is small, $\sin\theta \approx \theta$ and therefore the expression becomes

$$-m g l \theta = I\alpha$$

which makes angular acceleration directly proportional to θ, satisfying the definition of simple harmonic motion.

Self-Test 21

(1) The simple pendulum executes SHM when

(A) mass of the bob is very large

(B) the amplitude of oscillation is small

(C) length of the pendulum is much smaller than the amplitude

(D) string is massless

(2) The sum of KE and PE for the mass *m* attached to a spring with constant *k* oscillating with amplitude *A* is

(A) $\frac{1}{2} kA^2$

(B) kA

(C) variable

(D) $\frac{1}{2} m v^2$

Short Answer Questions 02

(1) Under what conditions the motion of a pendulum will not be a SHM?

(2) Show that the addition of kinetic and potential energy for a simple pendulum is always a constant.

(3) Show that the addition of kinetic and potential energy for a mass attached to spring is always a constant.

(4) Show that the addition of kinetic and potential energy for a body undergoing uniform circular motion is always a constant.

02-03: DAMPED OSCILLATIONS.

In real oscillators, friction or damping slows the motion of the system. Due to frictional force, the velocity decreases in proportion to the acting frictional force. While simple harmonic motion oscillates with only the restoring force acting on the system, damped harmonic motion experiences friction. In many vibrating systems the frictional force F_f can be modeled as being proportional to the velocity v of the object: $F_f = -cv$, where c is called the viscous damping coefficient.

Balance of forces (Newton's second law) for damped harmonic oscillators is then.

$$F = F_{\text{ext}} - kx - c\frac{\mathrm{d}x}{\mathrm{d}t} = m\frac{\mathrm{d}^2x}{\mathrm{d}t^2}.$$

When no external forces are present (i.e. when $F_{ext} = 0$), this can be rewritten into the form:

$$\frac{\mathrm{d}^2x}{\mathrm{d}t^2} + 2\zeta\omega_0\frac{\mathrm{d}x}{\mathrm{d}t} + \omega_0^2 x = 0,$$

Where

$$\omega_0 = \sqrt{\frac{k}{m}}$$

is called the "undamped angular frequency of the oscillator",

$$\zeta = \frac{c}{2\sqrt{mk}}$$

is called the "damping ratio".

The value of the damping ratio ζ critically determines the behavior of the system. A damped harmonic oscillator can be:

- Overdamped ($\zeta > 1$): The system returns (exponentially decays) to steady state without oscillating. Larger values of the damping ratio ζ return to equilibrium more slowly.
- Critically damped ($\zeta = 1$): The system returns to steady state as quickly as possible without oscillating (although overshoot can occur). This is often desired for the damping of systems such as doors.
- Underdamped ($\zeta < 1$): The system oscillates (with a slightly different frequency than the undamped case) with the amplitude gradually decreasing to zero. The

angular frequency of the underdamped harmonic oscillator is given by $\omega_1 = \omega_0\sqrt{1-\zeta^2}$,

- the exponential decay of the underdamped harmonic oscillator is given by $\lambda = \omega_0\zeta.$

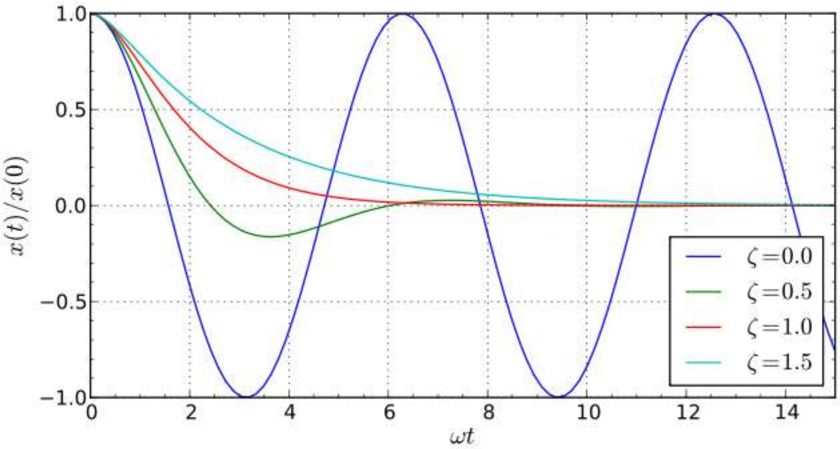

Fig 02-05: Dependence of system behavior on the value of the damping ratio ζ

The Q factor of a damped oscillator is defined as

$$Q = 2\pi \times \frac{\text{energy stored}}{\text{energy lost per cycle}}.$$

Let us see what this Q factor is

Q FACTOR

In physics and engineering the quality factor or Q factor is a dimensionless parameter that describes how underdamped an oscillator or resonator is, and characterizes a resonator's bandwidth relative to its centre frequency. Higher Q indicates a lower rate of energy loss relative to the stored energy of the resonator; the oscillations die out more slowly. A pendulum suspended from a high-quality bearing, oscillating in air, has a high Q, while a pendulum immersed in oil has a low one. Resonators with high quality factors have low damping, so that they ring or vibrate longer.

DEFINITIONS OF Q FACTOR

In the context of resonators, there are two common definitions for Q, which aren't necessarily equivalent. They become approximately equivalent as Q becomes larger, meaning the resonator becomes less damped. One of these definitions is the frequency-to-bandwidth ratio of the resonator:

$$Q \stackrel{\text{def}}{=} \frac{f_r}{\Delta f} = \frac{\omega_r}{\Delta \omega},$$

where f_r is the resonant frequency, Δf is the resonance width or full width at half maximum (FWHM) i.e. the bandwidth over which the power of vibration is greater than half the power at the resonant frequency, $\omega_r = 2\pi f_r$ is the angular resonant frequency, and $\Delta\omega$ is the angular half-power bandwidth.

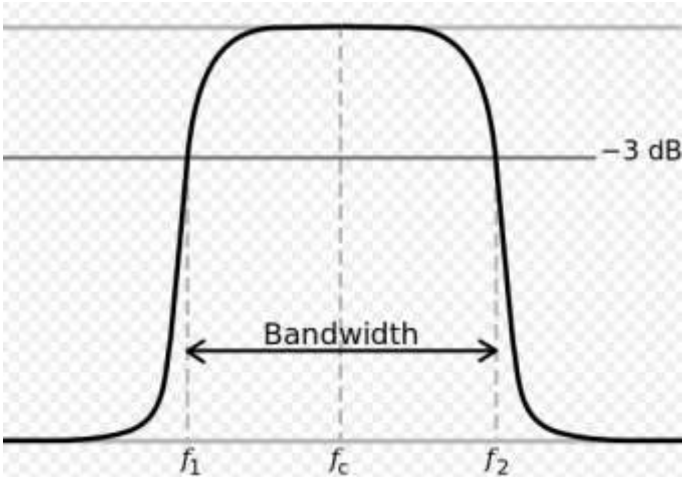

Fig 02-06: The Q factor is the ratio of f_c (the central frequency or resonance frequency) to bandwidth f_2-f_1 (The amplitude of oscillation fall to half the maximum power at f_2 and f_1. The -3dB indicates loss of half the power. The higher the Q, the narrower and 'sharper' the peak is.

The other common equivalent definition for Q is the ratio of the energy stored in the oscillating resonator to the energy dissipated per cycle by damping processes.

(The expression 'def' above the equality sign means that the equality is by definition, and hence not to be challenged, as it is not a derived expression.)

$$Q \stackrel{\text{def}}{=} 2\pi \times \frac{\text{energy stored}}{\text{energy dissipated per cycle}} = 2\pi f_r \times \frac{\text{energy stored}}{\text{power loss}}.$$

SELF-TEST 22

(1) The Q factor for an oscillator which stores 1.2 mW of power while dissipating 0.6mW in every cycle is given by

(A) 0.50

(B) 2

(C) 4 π

(D) 2 π

(2) The Q factor for a resonator of bandwidth 2MHz and resonating frequency 200MHz will be

(A) 1/100

(B) 100 π

(C) π/100

(D) 2 π/100

(3) The damping ratio for critically damped oscillations is

(A) More than 1

(B) equal to 1

(C) less than 1

(D) equal to 2 π

Short Answer Questions 03

(1) Define the Q factor for a damped oscillator.

(2) Derive the differential equation for a damped oscillator and solve it.

(3) What is meant by over-damped oscillation? Under what conditions the motion becomes over-damped?

(4) What is meant by critically damped oscillation? Under what conditions the motion becomes critically damped?

(5) What is meant by under damped oscillation? Under what conditions the motion becomes under damped?

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COURSE COMPANION WEBSITE

Visit Here for Course Companion website for this course:

THEORY CREDIT 04

UNIT 04-01: ELASTICITY

LEARNING OBJECTIVES

After successful completion of this unit, you will be able to

- Explain the definition of stress and strain under various types of deformation of shape and size due to applied load
- State the Hooke's law
- Discuss the features of stress-strain diagram over the entire range of loads up to point of rupture.
- Elaborate the three types of elasticity and define Young's modulus, bulk modulus, modulus of sheer and Poisson's ratio
- Derive the expression for relations between three elastic constants
- Derive the expression for work done per unit volume for the three types of strains
- Derive the expression for twisting couple of a cylinder and thereby for strain energy in twist (torsion)
- Explain the methods for determining rigidity modulus by static torsion and using torsion pendulum.
- Explain the Searle's experiment

INTRODUCTION

What is the most important feature of the material rubber? Well you will say, its flexibility: ability to stretch it. I would say that ability to stretch also exists in the dough of flour or the chewed gum. The rubber, you will then, correct yourself has the ability to stretch easily and then get back its form when the applied force is removed.

This property of most of the solids (and also of fluids) is called elasticity. We use elastic property of rubber when we use so-called "elastics" used in holding fabrics securely against our body (in track suits and undergarments) we use them in rubber bands and so on.

Elasticity is a very important topic in study of the branch of physics called 'Property of Material'. When we use a material in construction, one of the important aspect of the material is how strong it is and whether it will bend with the application of load. This is studied under ‘bending moments’ which is closely related to study of elasticity. (It is however not in our course of study here)

So, prepare yourself to study this important topic of Elasticity.

01-01: Hooke’s law - Stress-strain diagram - Elastic moduli-

Elasticity

In Physics, elasticity (from Greek ἐλαστός "ductible") is the ability of a body to resist a distorting influence and to return to its original size and shape when that influence or force is removed. Solid objects will deform when adequate forces are applied to them. If the material is elastic, the object will return to its initial shape and size when these forces are removed.

The physical reasons for elastic behavior can be quite different for different materials. In metals, the atomic lattice changes size and shape when forces are applied (energy is added to the system). When forces are removed, the lattice goes back to the original lower energy state. For rubbers and other polymers, elasticity is caused by the stretching of polymer chains when forces are applied.

Perfect elasticity is an approximation of the real world. The most elastic body in modern science found is quartz fibre which is not even a perfect elastic body. Perfect elastic body is an ideal concept only. Most materials which possess elasticity in practice remain purely elastic only up to very small deformations. In engineering, the amount of elasticity of a material is determined by two types of material parameter. The first type of material parameter is called a modulus, which measures the amount of force per unit area needed to achieve a given amount of deformation. The SI unit of modulus is the pascal (Pa). A higher modulus typically indicates that the material is harder to deform. The second type of parameter measures the elastic limit, the maximum stress that can arise in a material before the onset of permanent deformation. Its SI unit is also pascal (Pa).

When describing the relative elasticities of two materials, both the modulus and the elastic limit have to be considered. Rubbers typically have a low modulus and tend to stretch a lot (that is, they have a high elastic limit) and so appear more elastic than metals (high modulus and low elastic limit) in everyday experience. Of two rubber materials with the same elastic limit, the one with a lower modulus will appear to be more elastic, which is however not correct.

HOOKE'S LAW

Hooke's law is a principle of physics that states that the force (F) needed to extend or compress a spring by some distance x scales linearly with respect to that of distance. That is: $F_s = k\,x$, where k is a constant factor characteristic of the spring: its stiffness and x is small compared to the total possible deformation of the spring. The law is named after 17th-century British physicist Robert Hooke. He first stated the law in 1676 as a Latin anagram. He published the solution of his anagram in 1678 as: ut tensio, sic vis ("as the extension, so the force" or "the extension is proportional to the force"). Hooke states in the 1678 work that he was aware of the law already in 1660.

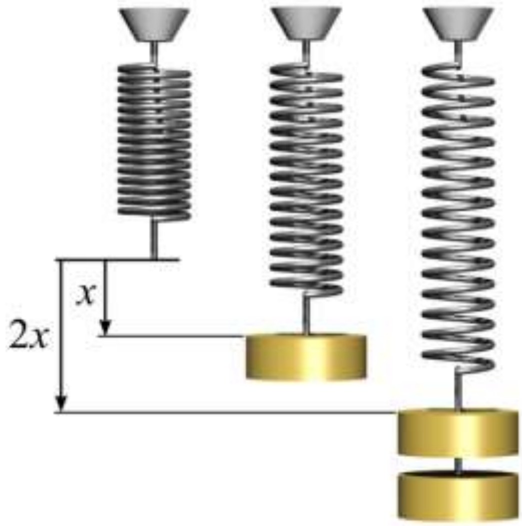

Fig 01-01: Hooke's law: the force is proportional to the extension

Hooke's equation holds (to some extent) in many other situations where an elastic body is deformed, such as wind blowing on a tall building, a musician plucking a string of a guitar, and the filling of a party balloon. An elastic body or material for which this equation can be assumed is said to be linear-elastic or Hookean.

Hooke's law is only a first-order linear approximation to the real response of springs and other elastic bodies to applied forces. It must eventually fail once the forces exceed some limit, since no material can be compressed beyond a certain minimum size, or stretched beyond a maximum size, without some permanent deformation or change of state. Many materials will noticeably deviate from Hooke's law well before those elastic limits are reached.

On the other hand, Hooke's law is an accurate approximation for most solid bodies, as long as the forces and deformations are small enough. For this reason, Hooke's law is extensively used in all branches of science and engineering, and is the foundation of many disciplines such as seismology, molecular mechanics and acoustics. It is also the fundamental principle behind the spring scale, the manometer, and the balance wheel of the mechanical clock.

The modern theory of elasticity generalizes Hooke's law to say that the strain (deformation) of an elastic object or material is proportional to the stress applied to it. However, since general stresses and strains may have multiple independent components, the "proportionality factor" may no longer be just a single real number, but rather a linear map (a tensor) that can be represented by a matrix of real numbers.

In this general form, Hooke's law makes it possible to deduce the relation between strain and stress for complex objects in terms of intrinsic properties of the materials it is made of. For example, one can deduce that a homogeneous rod with uniform cross section will behave like a simple spring when stretched, with a stiffness k directly proportional to its cross-section area and inversely proportional to its length.

Stress and Strain

In continuum mechanics, *stress* is a physical quantity that expresses the internal forces that neighboring particles of a continuous material exert on each other, while *strain* is the measure of the deformation of the material.

For example, when a solid vertical bar is supporting an overhead weight, each particle in the bar pushes on the particles immediately below it. When a liquid is in a closed container under pressure, each particle gets pushed against by all the surrounding particles. The container walls and the pressure-inducing surface (such as a piston) push against them in (Newtonian) reaction. These macroscopic forces are actually the net result of a very large number of intermolecular forces and collisions between the particles in those molecules. Stress is frequently represented by a lowercase Greek letter sigma (σ).

Strain inside a material may arise by various mechanisms, such as stress applied by external forces to the bulk material (like gravity) or to its surface (like contact forces, external pressure, or friction). Any strain (deformation) of a solid material generates an internal elastic stress, analogous to the reaction force of a spring, that tends to restore the material to its original non-deformed state. In liquids and gases, only deformations that change the volume generate persistent elastic stress. However, if the deformation is gradually changing with time, even in fluids there will usually be some viscous stress, opposing that change. Elastic and viscous stresses are usually combined under the name mechanical stress.

Mechanic stress

Significant stress may exist even when deformation is negligible or non-existent (a common assumption when modeling the flow of water). Stress may exist in the absence of external forces; such built-in stress is important, for example, in pre-stressed concrete and tempered glass. Stress may also be imposed on a material without the application of net forces, for example by changes in temperature or chemical composition, or by external electromagnetic fields (as in piezoelectric and magnetostrictive materials).

LOAD

In the present context, load means the combination of external forces acting on a body and its effect is to change the form or dimension of the body. I can give a number of examples to make it clear to you: First example: the weight of the body itself together with other bodies connected to it makes up the load as these tend to change the shape or form of the body. Another example, in case of rotating bodies like pulleys and wheels, the centrifugal forces tend to change the shape of the body. Similarly, unequal expansion of parts of bodies like bimetallic strip used in electrical iron ('press') to control the temperature deforms the shape of the strip and hence is an example of load. Load is thus a deforming force.

STRESS

As a result of load (deforming force) applied to a body, *forces of reaction* internally start acting their roles. Due to relative displacement of molecules these forces try to get back the body to its original shape and size. These recovering forces per unit area set up inside the body are called *stress*. It is a distributed force. It is therefore measured in the same manner as we measure fluid pressure: in terms of deforming force (load) per unit area.

Hence if the force F is applied at an angle θ to the surface of area A of the body, the component $F\cos(\theta)$ normal (perpendicular) to the plane of the surface can be taken as deforming force and such stress $F\cos(\theta)/A$ is called *normal stress*. The component which is in the direction tangential to the surface (same direction as the surface) $F\sin(\theta)/A$ is called *tangential* or *sheering* stress. The normal stress tends to change the length of a wire or cylinder while sheering stress on a body tends to change its shape.

Stress may be compressive if the body is subjected to external axial thrust and it may be tensile (causing tension) if the body is subjected to axial pull across any section perpendicular to its length.

Since stress is force per unit area, its unit is Nm^{-1} or Pascal (Pa).

STRAIN

The change produced in the dimensions of a body under a system of forces or couples in equilibrium is called strain.It is measured according to the nature of deformation as

Change in length per unit length (dl/L) (dl being change in length and L being original length) in case of linear strain

Change in volume per unit volume (dV/V) (dV being change in volume and V being original volume) in case of volume strain

Change in angle of axis of a plane in case of a sheer

Strain is a dimensionless quantity.In case of a perfectly elastic body

(i) Strain is always same for a given stress. Suppose, I take a bar of perfectly elastic body of length 2 m and subject it to a stress of 5 N/m^2. After two days if I observe, I find its length has changed by 120 microns. If I repeat the same experiment today or tomorrow, I will observe the same result: 120 microns change in the length with 5 Pa load.

(ii) Strain vanishes completely when load is removed. In terms of my earlier experiment, when I removed the load of 5 Pa, the bar got back to its original size losing 120 microns which it had gained.

(iii) The stress required to maintain a given strain is the same.

STRESS STRAIN DIAGRAM

The relationship between the stress and strain that a particular material displays is known as that particular material's **stress–strain curve**. It is unique for each material and is found by recording the amount of deformation (strain) at distinct intervals of tensile or compressive loading (stress). These curves reveal many of the properties of a material.

Stress–strain curves of various materials vary widely, and different tensile tests conducted on the same material yield different results, depending upon the temperature of the specimen and the speed of the loading. It is possible, however, to distinguish some common characteristics among the stress–strain curves of various groups of materials and, on this basis, to divide materials into two broad categories; namely, the ductile materials and the brittle materials.

The Figure shows the stress-strain curve for a typical ductile material.

The strain is applied on the sample with increasing value of strain in this diagram. Initial part OA of the curve shows stress to be proportional to strain according to the Hooke’s law. If the strain is removed, the sample gets back its original shape and form shown by point O (zero load). The sample is perfectly elastic up to stress represented by A. The point A till which the sample is elastic as characterized by the three properties described earlier, is called elastic limit. In some cases the Hooke's law is obeyed little below the elastic limit.

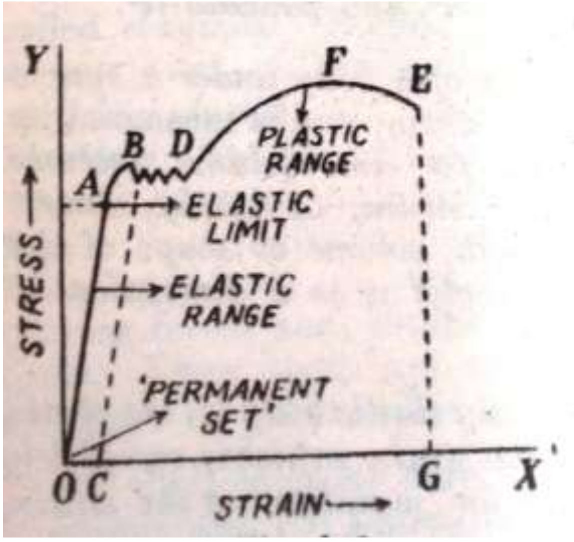

Fig 01-02: Typical stress-strain curve for a ductile material

The stress at the elastic limit is called elastic strength of the material. The strain produced till elastic limit is usually as small as 10^{-3} of the original length of the sample.

After the elastic limit has been crossed the strain changes more rapidly with the same stress in part AB. The sample is now a combination of elastic and plastic. The unloading of the sample does not show sample getting back its shape and size in region AB. If you have applied a stress as at point B and slowly remove the stress, it follows the dotted path BC, so that there remains a residual strain OC in it. This is called '***permanent set***' acquired by the sample.

Beyond point B, there is an erratic increase in strain up to point D. Thus the portion BD is shown as irregular wavy nature. Stress corresponding to D is less than that at B. The point B where the large increase in strain (in extension) starts is called '***yield point'***.. The stress at yield point is called yielding stress. Some people call B commercial elastic limit and sometimes points B and D are called as *upper and lower yield points* respectively.

The yielding stops at D and if you keep increasing stress, you find the cross section of the wire decreasing uniformly with increasing strain of the sample. The part DF thus shows that sample has become plastic, the sheering stress dominating the simple tensile stress. The maximum load or force to which the sample is subjected divided by the original cross sectional area is called ultimate strength of the sample wire and is also sometimes called breaking stress.

Beyond point F, the sample behaves as if it were literally flowing down. The extension of the wire goes on increasing beyond F without any addition to the load, indeed even if the load be reduced a little. This is because of faster rate of decrease of its cross sectional area at some section of its length where a local constriction called neck or waist begins to develop. Hence even if the load is not increased, the load per unit area (i.e. stress) is considerably greater there, bringing about a corresponding increase in strain (extension) in wire. The load is therefore decreased, i.e., stress is reduced at this stage. The sample finally ruptures or snaps or breaks at E, which represents the breaking point for it.

THREE TYPES OF ELASTICITY

Corresponding to three types of strain, there are three types of elasticity

(1) Elasticity of Length (linear elasticity) called Young's modulus corresponding to linear or tensile strain.

(2) Bulk Modulus corresponding to elasticity of volume

(3) Modulus of Rigidity or Shear Modulus corresponding to shear strain (causing change in shape of body)

YOUNG'S MODULUS:

When deforming force is applied to the body only along a particular direction the change per unit length in that direction is called longitudinal or linear or elongation strain. The force applied per unit area of cross section is called longitudinal or linear stress. The ratio of longitudinal stress to longitudinal strain within the elastic limit is called Young's modulus Y.

Let F be the force applied normally to the cross section area a. The Stress is then F/a. If it produces a change l of length in the original length L (before application of stress), strain is l/L.

The Young's modulus $Y = \frac{\frac{F}{a}}{\frac{l}{L}} = \frac{FL}{al}$

If l =L and a =1, then Y =F.

We can define Y as the force applied to a material of unit cross sectional area such that the length of the sample gets doubled. This is not a practical definition as the elastic limit allows only changes as small as 10^{-3}.

YOUNG'S MODULUS FOR RUBBER IS MUCH LESS THAN THAT OF STEEL

You may be confused to know that the Y for rubber is around one-fiftieth that of steel. You know that it takes much less force to deform a piece of rubber thread in comparison to that of a steel wire with same diameter.

In the definition of Y, we took the ratio of stress to strain. For rubber much less force is required to cause the same deformation as in steel. Hence Y for rubber is around $1/50^{th}$ to that of the steel.

Second point about comparison between rubber and steel is that rubber has a very wide range of elasticity. In case of crystalline material like steel, the body can be stretched to less than 1% of its original length before reaching the elastic limit. On the other hand, a piece of rubber can be stretched to as much as 80% of the original length before reaching the elastic limit. The high extendibility of rubber is due to the molecules containing some 4000 molecules of isoprene (C_5H_8) whose 20,000 C atoms spreading out in chain make it very long and thin (about 1/4000 mm in length).

Bulk rubber is like intertwined mass of long, wriggling snakes. Its molecules, like snakes, tend to uncoil when stretched and get coiled up again when stretching force is removed.

BULK MODULUS

Here the force is applied normally and uniformly to the whole of the surface of a body so that, even though there is a change in volume, there is no change of shape. In other words, we have a change of scale of the coordinates in the system or the body. Force applied per unit area gives the stress and change in unit volume gives the strain. The ratio of stress to strain gives Bulk Modulus.

Thus if force F is applied to the surface area 'a' in uniform and normal manner, and change of volume v is observed to the orginal (un-stressed) volume of V, stress will be equal to F/a = P (pressure) and strain will be v/V and bulk modulus K will be equal to:

$$K = \frac{\frac{F}{a}}{\frac{v}{V}} = \frac{FV}{av} = \frac{PV}{v}$$

We can consider infinitesimal change in volume dV corresponding to change of pressure dP, to define K as:

K =dP.V/dV

Let us see some of the features of K.

FLUIDS ONLY HAVE K AS ELASTIC MODULUS

Since fluids (gas and liquid state of matter) can be subjected to only hydrostatic pressure as stress, the only elastic properties which can be defined for them is bulk modulus. To understand this, ask yourself, can you have 'wires of gases' so that Y can be defined for it? No. Similarly can you have a sample of liquid or gas which can be subjected to shear stress? No!

COMPRESSIBILITY AND INCOMPRESSIBILITY

Bulk modulus is sometimes referred to as incompressibility of the material. This is because high value of K implies that the material is difficult to be compressed. Why? Because if the given material sample is difficult to compress it means you need more pressure to achieve the same change in volume in comparison to another material which is easier to compress. The value of K for incompressible material therefore would be more. Get back to my discussion on Young's modulus for rubber and for the steel for analogy.

We define another physical quantity or parameter called Compressibility of a material which is defined as reciprocal of bulk modulus K.

MODULUS OF RIGIDITY

In this case there is a change in shape and no change in volume. The mechanism of deformation is similar to that of a deck of cards in which cards at the top slide over those below them. The shape of the deck changes from rectangular parallelepiped to an angular one. In the given sample subjected to a shear stress, there is a movement of layers of body one over the other to produce similar effect.

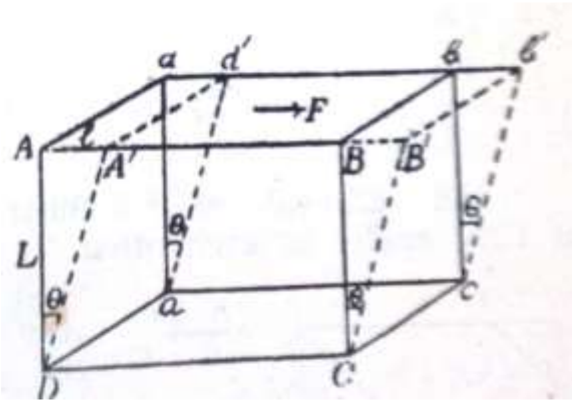

Fig 01-03: Shear stress and Shear

Consider a rectangular cube ABCDacbd, with its lower face aDCc fixed and a tangential force F is applied to upper face abBA in the direction shown in Figure 01-03. A couple is produced by this force with an equal and opposite force coming into play at the lower fixed face, making layers parallel to the two faces move over one another. This means that A shift to A', d to d', B to B' and b to b'. The new object A'B'CDacb'd' shows that the angle ADA' is equal to BCB' which is further equal to angle bcb' and angle dad'. Let us call this angle θ .That is, deformation produced is the angle θ .

The face ABCD is said to be sheared through an angle θ. This angle θ in radian gives the strain or shear strain or angle of shear.

$$\theta = l/L$$

Here l is the displacement AA' and L is the length AD of the cube. Angle θ gives the relative displacement of the plane.

If L =1 (unity) then $\theta = l$ = Relative displacement of plane ABbd.

Hence shear strain can be defined as relative displacement of the planes which are unit distance apart.

The tangential stress is equal to Force F divided by area of face ABbd =F/a.

The Modulus of rigidity η is defined as ratio of tangential stress to shear strain.

$$\eta = \frac{F/a}{\theta} = \frac{F/a}{l/L} = \frac{FL}{al}$$

This formula for Modulus of rigidity looks similar to that for Young's modulus Y. There are two differences: First one being that the force applied is parallel to the face of surface in case of η , while it is perpendicular to the surface of sample in case of Y. the second difference is that change in length l is at right angle to the length L in case of η, while in case of Y it is in the same direction.

In order to get an idea about the real world situation, I am listing in the following table the relative values of elastic moduli of various substances taken from College Physics (OpenStax book available for free at http://cnx.org/content/col11406/1.9)

Material	Young's modulus (tension–compression) *Y* (10^9 N/m^2)	Shear modulus *S* (10^9 N/m^2)	Bulk modulus *B* (10^9 N/m^2)
Aluminum	70	25	75
Bone – tension	16	80	8
Bone – compression	9		
Brass	90	35	75
Brick	15		
Concrete	20		
Glass	70	20	30
Granite	45	20	45
Hair (human)	10		
Hardwood	15	10	
Iron, cast	100	40	90
Lead	16	5	50
Marble	60	20	70
Nylon	5		
Polystyrene	3		
Silk	6		
Spider thread	3		
Steel	210	80	130
Tendon	1		
Acetone			0.7
Ethanol			0.9
Glycerin			4.5
Mercury			25
Water			2.2

Solved Problems

Example 01: The Stretch of a Long Cable

Suspension cables are used to carry gondolas at ski resorts. (See Figure 01-04) Consider a suspension cable that includes an unsupported span of 3 km. Calculate the amount of stretch in the steel cable. Assume that the cable has a diameter of 5.6 cm and the maximum tension it can withstand is 3.0×10^6 N

Fig 01-04: Gondola lift (cable-car) carrying tourists on ski resorts

Strategy

The force is equal to the maximum tension, or $F = 3.0\times10^6$ N . The cross-sectional area is $\pi r^2 = 2.46\times10^{-3}$ m^2 . The equation $\Delta L = \frac{1}{Y}\frac{F}{A}L_0$ can be used to find the change in length.

Solution

All quantities are known. Thus, simply plug the quantities in the above equation; you get answer as 18m.

Discussion

This is quite a stretch, but only about 0.6% of the unsupported length. Effects of temperature upon length might be important.

Example (2) Calculating Deformation: How Much Does Your Leg Shorten When You Stand on It?

Calculate the change in length of the upper leg bone (the femur) when a 70.0 kg man supports 62.0 kg of his mass on it, assuming the bone to be equivalent to a uniform rod that is 40.0 cm long and 2.00 cm in radius.

Strategy

The force is equal to the weight supported, or

$$F = mg = (62.0\text{ kg})\left(9.80\text{ m/s}^2\right) = 607.6\text{ N},$$

and the cross-sectional area is $\pi r^2 = 1.257\times10^{-3}$ m^2. The equation $\Delta L = \frac{1}{Y}\frac{F}{A}L_0$ can be used to find the change in length.

Solution

All quantities except ΔL are known. Note that the compression value for Young's modulus for bone must be used here.

Thus,

$$\begin{aligned}\Delta L &= \left(\frac{1}{9\times10^{9}\text{ N/m}^2}\right)\left(\frac{607.6\text{ N}}{1.257\times10^{-3}\text{ m}^2}\right)(0.400\text{ m})\\ &= 2\times10^{-5}\text{ m.}\end{aligned}$$

Discussion

This small change in length seems reasonable, consistent with our experience that bones are rigid. In fact, even the rather large forces encountered during strenuous physical activity do not compress or bend bones by large amounts. Although bone is rigid compared with fat or muscle, several of the substances listed in Table have larger values of Young's modulus Y. In other words, they are more rigid.

SELF-TEST 23

(1) What is Young's modulus ?

(A) Strain / Stress

(B) Change in length / original length

(C) Stress / change in length

(D) Longitudinal stress / longitudinal strain

(2) What is the property of a body by virtue of which it tends to regain its original size and shape by the removal of applied force is called?

(A) Elasticity

(B) Plasticity

(C) Deformation

(D)Strain

(3) Which of the following comes under the category of longitudinal stress?

(A) Tensile stress

(B) Compression stress

(C) Shearing stress

(D) Both (A) and (B)

(4) When the length of a material increases due to applied stress, what will happen to its volume?

(A) decreases

(B) Remains constant

(C) increases

(D) no dependence

(5) The liquid and gases can undergo only ….. strain?

(A)Tensile

(B)Torsion

(C)Volume

(D) shear

(6) What happen when bodies are deformed beyond elastic limit?

(A) Hooke's law is not obeyed

(B) the body does not get back to its original size and shape after the load is removed

(C) the body becomes plastic

(D) all of the above

(7) The young's modulus of a wire of radius R and length L is Y N/m^2. If the radius and length are changed to 2R and 4L respectively, then its young's modulus will be?

(A)8Y

(B)0

(C) Y

(D) 2Y

Short Answer Questions 01

(1) Explain the characteristics of a perfectly elastic body

(2) Draw the stress-strain curve for a typical metal wire and explain the various parts of the diagram from zero strain to point of rupture.

(3) What is property of elasticity? Is it only applicable to solids?

(4) What type(s) of elasticity exist for liquid and gases?

01-02: Relation between elastic constants and Work done in stretching and in twisting a wire

Relation between elastic constants

The elastic constants show how much the sample gets deformed when a deformative load is applied. They differ in the manifestation of the deformation: Y shows how much length is changed when tensile stress is introduced while η shows how much shear is produced with shearing stress. Since elasticity is the property of a material, modulus of elasticity are interdependent.

We will derive the expression of relation for the three moduli now. For this we consider a sample as unit cube shown in Figure below.

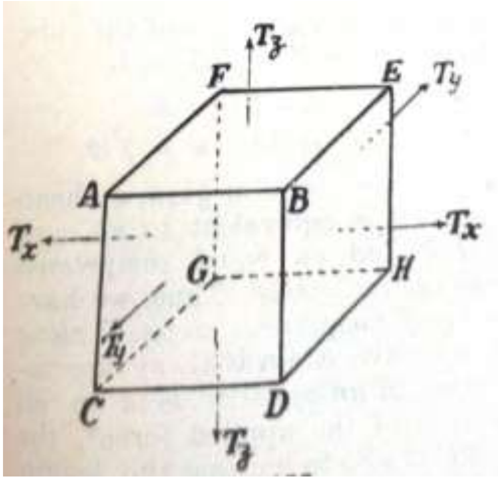

Fig 01-05: Cube of the elastic material

Let ABDCGHEF be unit cube and forces Tx, Ty and Tz act perpendicular to the faces as shown in Fig 01-05.

Let α be increase per unit length in the direction of force, β be the contraction produced per unit length per unit tension in direction perpendicular to the force.

Elongation produced in edges AB, BE and BD will be αTx, αTy, and αTz.

Contraction produced perpendicular to edges AB, BE and BD will be βTx, β Ty, βTz.

Lengths of Edges will be as follows:

New(AB) = $1+ \alpha Tx- \beta Ty- \beta Tz$

New (BE) = $1- \alpha Tx+ \beta Ty- \beta Tz$

New(BD) =$1- \alpha Tx-\beta Ty+ \beta Tz$

Volume of the cube will now be

$(1+ \alpha Tx- \beta Ty- \beta Tz)(1- \alpha Tx+ \beta Ty- \beta Tz)(1- \alpha Tx-\beta Ty+ \beta Tz)$

$= 1 + (\alpha-2 \beta)(Tx+Ty+Tz)$

We have neglected terms in squares and products of β and α. Let us consider the case when Tx=Ty=Tz = T. Then

New volume = $1 + 3T (\alpha-2 \beta)$

Increase in volume of cube = $3 T (\alpha-2 \beta)$

If instead of tension T if we had applied pressure P to compress the cube, reduction in volume will be $3 P (\alpha-2 \beta)$.

This allows us to express bulk modulus as

$$K = \frac{stress}{volume\ strain} = \frac{P}{3P(\alpha - 2\,\beta)}$$

$$K = 1/3(\alpha - 2\,\beta)$$

(01-01)

This expression is very useful in finding the relationship among the moduli.

We can similarly find expression for Y by considering unit cube acted upon by unit tension along one edge, with extension produced as α. Then, clearly,

Stress = 1. And linear strain = $\alpha/1 = \alpha$

(01-02)

This is the second expression which will be used in finding the relation among moduli of elasticity.

Now let us consider the shear. For this I refer to the figure 01-06

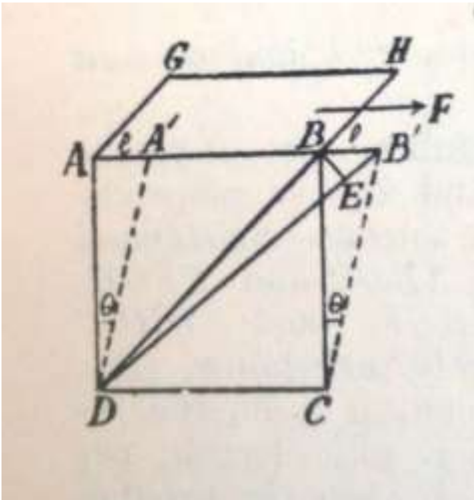

Fig 01-06: Cube subject to shear

The top face ABHG of the cube is sheared by shearing force F relative to the bottom face. It makes A go to A' and B shifted to B'. The angle ADA' is equal to angle BCB' =θ.

Tensile stress T = $\frac{Force}{Area\ of\ ABHG} = \frac{F}{L^2}$

Let displacement AA' =BB' =l

Then shearing strain = l/L = θ

Modulus of Rigidity $\eta = T/\,\theta$

It can be shown that shearing stress along AB is equivalent to equal linear tensile stress along diagonal DB and an equal compressive stress along diagonal AC, acting at right angles to each other.

If α and β are longitudinal and lateral strains per unit stresses respectively, then

Extension of DB due to tensile stress along it = α. T. DB

Extension of DB due to compressive stress along AC = β. T. DB

Total extension in length of DB = DB.T.($\alpha + \beta$) $= L\sqrt{2}\,T(\alpha + \beta)$

We have used the fact that diagonal DB= $L\sqrt{2}$, where L is the edge of the cube.

We drop a perpendicular BE from B on to DB'. Then the increase in length of DB is approximately equal to EB'. Also note that ∠BB'E =45° and that $\cos(45°) = 1/\sqrt{2}$

EB' = BB'cos (45°)= $l\cos(45°) = l/\sqrt{2}$

Hence $L\sqrt{2}\,T(\alpha + \beta) = l/\sqrt{2}$

Or, $T\,.\frac{L}{l} = \frac{1}{2(\alpha+\beta)}$

Or $\frac{T}{\theta} = \frac{1}{2(\alpha+\beta)}$

[Since, $\frac{L}{l} = \theta$]

But we know that $\frac{T}{\theta} = \eta$ (Modulus of Rigidity of material)

$$\eta = \frac{1}{2(\alpha + \beta)}$$

(01-03)

Now that we have expressions for all three moduli of elasticity viz Y, K and η in terms of tensile and compressive stains for a cube, we can proceed to prove the relation among Y, K and n.

From (01-01),

$$K = 1/3(\alpha - 2\,\beta)$$

i.e. $\alpha - 2\,\beta = \frac{1}{3K}$

(01-04)

From (01-03) I can write:

$$\alpha + \beta = \frac{1}{2\eta}$$

(01-05)

Subtracting equation 01-04 from 01-05, α gets cancelled, leaving only β :

$$3\beta = \frac{1}{2\eta} = \frac{1}{3K}$$

Simplifying:

$$\beta = \frac{3K - 2\eta}{18\eta K}$$

(01-06)

If I multiply (01-03) by 2 and add it to (01-01) I will get:

$$3\alpha = \frac{1}{\eta} + \frac{1}{3K} = \frac{3K + \eta}{3\eta K}$$

$\alpha = \frac{3K+\eta}{9K\eta}$

From (01-02) I can say that $\alpha = \frac{1}{Y}$

$$\frac{1}{Y} = \frac{3K + \eta}{9K\eta}$$

Or $Y = \frac{9K\eta}{3K+\eta}$

(01-07)

This is the required expression, which shows relation between Y, K and η.

However, this expression is useful if you want to calculate Y. It may become messy if you wish to calculate η or K. For this I will simplify (01-07) to a form which will be more useful:

If I take reciprocal of the equation (01-07), I can write (01-07) as:

$$\frac{1}{Y} = \frac{3K + \eta}{9K\eta}$$

$$\frac{1}{Y} = \frac{1}{3\eta} + \frac{1}{9K}$$

I am further simplifying by multiplying this equation by 9;

$$\frac{9}{Y} = \frac{3}{\eta} + \frac{1}{K}$$

(01-08)

This is the most important form of the expression showing relation between Y,K and η.

POISSON'S RATIO

When I apply the stress on a wire, it gets elongated in the direction of the stress. If the dimensions in other directions had remained the same, it would mean that the total volume has increased and therefore density of the sample has decreased. Such thing does not happen very dramatically and when one dimension increases, there is a corresponding decrease in the other dimensions. It has been however found that volume gets slightly increased.

Thus when we stretch a wire, there is increase in its length and slight decrease in its cross section. Wire becomes longer and thinner. In other words linear strain is accompanied by transverse or lateral strains of opposite kind. Thus when you compress the material, cross section increases. The strain along the direction of stress is called primary strain and that in the directions right angle to it is called secondary strain.

Suppose I apply the tensile (tension) stress along the X-axis of the sample, I will observe increase in the dimension along X axis (corresponding to primary strain) and I will observe decrease in dimensions along Y and Z axis (which correspond to the secondary strain)

Within the elastic limit, the secondary strain is proportional to the primary strain. The constant of proportionality is called Poisson's ratio (σ) for the given material.

Thus, $\sigma = \frac{Secondary\ strain}{Primary\ strain}$

In case of a wire, subjected to a linear or tangential stress, we had denoted the linear strain by α and transverse strain by β. For this situation

$$\sigma = \frac{Secondary\ strain}{Primary\ strain} = \frac{\beta}{\alpha}$$

RELATION BETWEEN BULK MODULUS AND MODULUS OF RIGIDITY IN TERMS OF POISSON'S RATIO

We had derived the equation (01-01)

$$K = 1/3(\alpha - 2\,\beta)$$

From this I can write:

$$K = \frac{1}{3\alpha\left(1 - \frac{2\beta}{\alpha}\right)} = \frac{1}{3\alpha(1 - 2\sigma)} = \frac{Y}{3(1 - 2\sigma)}$$

Here I have made use of the definition of Poisson's ratio and also used (01-02) which gave Y in terms of primary strain α.

I may write it in the following form:

$$Y = 3K(1 - 2\sigma)$$

(01-09)

This is the relation between Young's modulus, Bulk Modulus and Poisson's ratio.

I will obtain one more relation between Y, η and σ. For that I use the relation (01-03)

$$\eta = \frac{1}{2(\alpha + \beta)}$$

$$\eta = \frac{1}{2\alpha\left(1 + \frac{\beta}{\alpha}\right)} = \frac{Y}{2(1 + \sigma)}$$

$$Y = 2\eta(1 + \sigma)$$

(01-10)

I will now derive an expression for σ in terms of K and η. I am making use of (01-09) and (01-10), by comparing RHS and eliminating Y.

$$3K(1 - 2\sigma) = 2\eta(1 + \sigma)$$

$$\sigma = \frac{3K - 2\eta}{6K + 2\eta}$$

(01-11)

WORK DONE IN STRETCHING AND WORK DONE IN TWISTING A WIRE -

In order to deform a body, work must be done by the applied force. The energy corresponding to this work gets stored in the material. This is a type of potential energy. It gets spent in the form of movement caused to get back to the previous size and shape and some energy is used in the heat generated, when we remove the applied stress. We call this energy as energy of strain.

I will now show you the derivation for work done per unit volume for three types of strain: (1) when wire is stretched (elongation) (2) when the material is subjected to pressure causing volume strain and (3) when shape of the body is changed due to shear.

WORK DONE IN ELONGATION STRAIN

If I apply a force F on one end of a wire (of length L) with other end fixed on a rigid surface, a small increase in length (l) will be observed.

I may divide the given wire into infinitely many sections each of length dL. The total change of length l will be the summation of changes in each of these infinitesimal segments. The work done in moving each of the infinitesimal segments will be *F.dL*.

I can write the expression for work done as

$$W = \int_0^l F.dL$$

The Young's modulus for the wire (with cross section area *a,* and change in length l) is given by

$$Y = \frac{F.L}{a.l}$$

which gives me:

$$F = \frac{Y.a.l}{L}$$

I will use it to derive expression for W as follows:

$$W = \int_0^l \frac{Y.a.l}{L} dL$$

$$W = \frac{Y.a}{L}\int_0^l l\, dL$$

$$W = \frac{Y.a.}{L}\frac{l^2}{2}$$

I will write this expression in a form which will be useful to me later

$$W = \frac{1}{2}.\frac{Y\,a\,l}{L}.l$$

I note that $\frac{Y\,a\,l}{L} = F$, Hence I write:

$$W = \frac{1}{2}F.l$$

(01-12)

Thus Work is half the product of force applied and elongation caused.

Work done per unit volume can be obtained by dividing W in (01-12) by volume (=L.a)

$$W' = \frac{1}{2}F\,l\,(\frac{1}{La})$$

I can rearrange it as follows:

$$W' = \frac{1}{2}\left(\frac{F}{a}\right).\left(\frac{l}{L}\right) = \frac{1}{2}.Stress.Strain$$

Thus work done per unit volume equals half the product of stress and strain.

It will be interesting to know that for any kind of stress (compressive, tensile or shear) work done per unit volume equals half the product of stress and strain.

WORK DONE PER UNIT VOLUME FOR VOLUME STRAIN

Let p be the stress applied. (Remember, pressure and volume stress are same). Then, over an area a, force applied is pa. The work done for a small movement dx in the direction of p is equal to p.a.dx. But, you may note that a.dx is dv (the change in volume). Hence work done for infinitesimal change in volume is p.dv.

For our specimen the total work done will be integration of p.dv over range 0 to v:

$W = \int_0^v p.dv$

Let us recall the definition of bulk modulus K for the original volume V

$$K = \frac{p.V}{v}$$

This gives me $p = \frac{Kv}{V}$. I put this in the expression for the work and simplify a bit to get

$$W = \frac{K}{V}\int_0^v vdv$$

$$W = \frac{K}{V}\frac{1}{2}\,v^2$$

$$W = \frac{1}{2}\cdot\frac{Kv}{V}.v$$

Identify $\frac{Kv}{V}$ as p, to get $W = \frac{1}{2}\,p\,v = \frac{1}{2}\,Stress.Chage\ in\ volume$

If we want to find work done per unit volume, we devide W by V

$$W' = \frac{1}{2}p\,\frac{v}{V} = \frac{1}{2}Stress\ \times Strain$$

WORK DONE PER UNIT VOLUME FOR SHEARING STRAIN

Let us have a cube of edge L, whose one face (denoted by DC) is fixed on a surface and stress is applied as tangential force on the opposite face (denoted by AB). This will cause each of the point on the face AB to move by a distance l. (AA’ = BB’ = l)

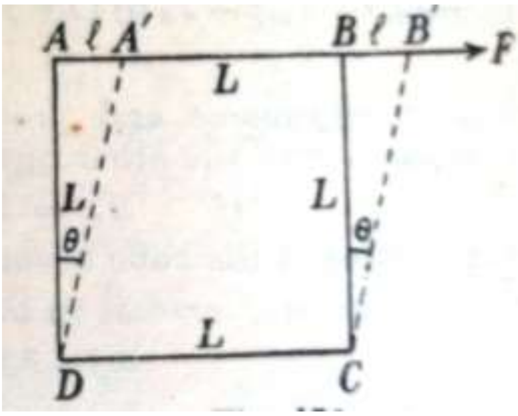

Fig 01-07: Shearing strain on a cube

Work done during a small displacement d l is equal to F.d l. Hence the work done for the whole of displacement from 0 to l is given by

$$W = \int_0^l F.dl$$

We recall the definition of Modulus of Rigidity η , $\eta = \frac{F}{a\theta}$. Or, $F = \eta\ a\,\theta$

We further apply the fact that $\theta = \frac{l}{\mathrm{L}}$ and, that $a = L^2$, so that $F = \eta\ L^2 \frac{l}{L} = \eta\ L\,l$

This gives me:

$$W = \eta L \int_0^l l\,dl \ = \eta\ L\frac{l^2}{2} = \frac{1}{2}\ \eta\ L\,l \times l \ = \frac{1}{2}\ F\ \times l \ = \frac{1}{2}\ Tangential\ Force\ \times Displacement$$

The work per unit volume (= a l) will be $W' = \frac{1}{2}\frac{F\,l}{a\,l} = \frac{1}{2}\ Stress \times Strain$

Thus we have proved that for any type of strain, work done per unit volume equals half the product of stress and strain.

Twisting couple on a cylinder

If we have a wire (i.e., a cylinder whose length is very large compared to its radius) which is clamped at one end to a fixed surface so that it cannot rotate about its axis at that fixed end and if we apply a torque or couple at the other free end, we say that the wire is under tension. You should recall that a couple refers to two parallel forces that are equal in magnitude, opposite in sense and do not share a line of action. Its effect is to create rotation without translation, or more generally without any acceleration of the centre of mass. In rigid body mechanics, force couples are free vectors, meaning their effects on a body are independent of the point of application. The quantity of the couple is equal to any one of the force multiplied by the perpendicular distance between the two forces.

Due to elasticity of the material of the wire, a restoring couple is developed on the fixed end of wire which is equal and opposite of the external couple at the free end.

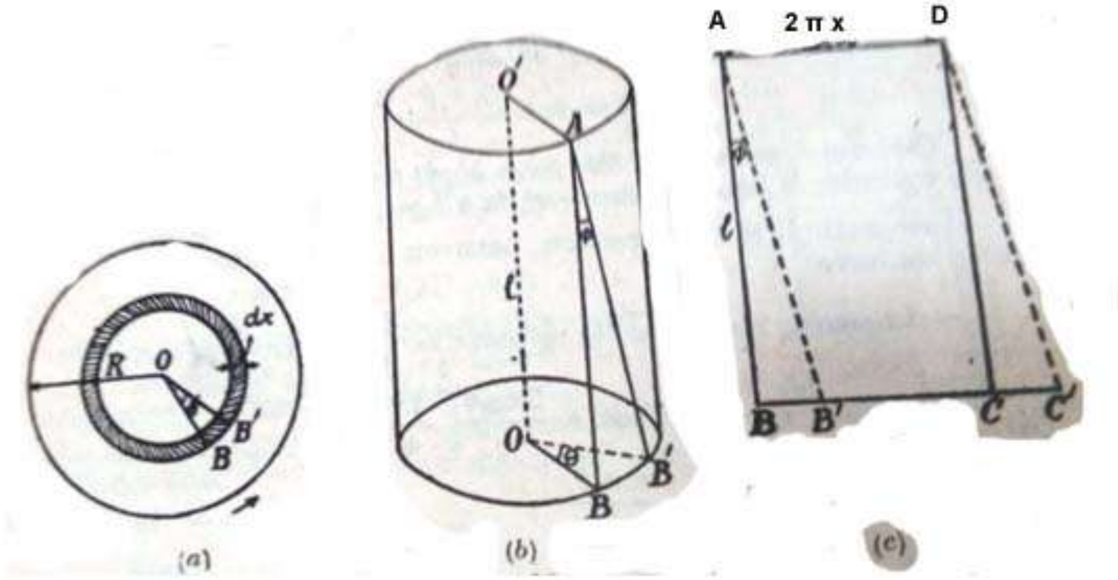

Fig 01-08: Twisting Couple applied to bottom surface of wire

The adjacent figure 01-08 illustrates the case of a wire with three aspects, labeled as (a), (b) and (c). In (a) part, you can see that the wire can be thought of as consisting of a number of hollow concentric cylinders around the axis of the wire. A typical of such hollow cylinder is shown here with radii x and x+dx. The torsion causes point B to move to B', making an angle BOB' denoted by θ.

In the second part (b), We can see that wire of length l with axis OO' is twisted due to application of external torque such that point B shifts to B' at the bottom surface making angle BOB' as θ. The top surface is fixed hence the point A does not get displaced due to the external torque. The angle BAB' is denoted by φ.

In the third part (c) we have flattened the surface so that it forms a rectangle ABCD under unstressed condition, with length l and width $2\pi x$. After the external stress is applied, the points B and C get shifted to B' and C', distorting the flattened surface from rectangle to a parallelepiped AB'C'D, with angle BAB' is denoted by φ.

This is to be noted that during twisting, there is no change in length or radius, thus no changes in the dimensions.

At the equilibrium, the twisting couple is equal and opposite of the restoring couple.

From (a) or (b) part of the figure, it is seen that

$BB' = l\,\varphi = x\,\theta$

The value of φ is greatest when x is greatest. Maximum strain is on the outermost part of cylinder and the least on the innermost, (zero at the axis).

Hence shearing stress is not uniform.

Angle of shear is the same for any *one* hollow cylinder. Thus, each and every point on line BC moves the same distance BB' or CC'.

However, the angle of shear is different for inner hollow cylinder and outer hollow cylinders. Angle of shear is maximum for outermost and the least for innermost.

Let us recall the definition of coefficient of rigidity

$$\eta = \frac{shearing\ stress}{strain\ or\ angle\ of\ shear} = \frac{F'}{\varphi}$$

Here F' denotes force per unit area of shearing surface. We have,

$$F' = \eta\varphi = \eta x\theta/l$$

The face area (crossection) of this hollow cylinder is 2πx.dx.

Hence, total shearing force is = area. F'

Total shearing force = 2πx.dx. $\frac{\eta x\theta}{l} = \frac{2\pi\eta\theta}{l}.x^2$dx

The moment of the force for the elementary surface about axis OO' is Total shearing force ($\frac{2\pi\eta\theta}{l}.x^2$dx) multiplied by perpendicular distance (x) =$\frac{2\pi\eta\theta}{l}.x^3$dx

The total moment of force can be found by integrating the contribution from elementary surface over limit x=0 and x=r.

Twisting couple for cylinder = $\int_0^r \frac{2\pi\eta\theta}{l}.x^3\mathrm{dx} = \frac{2\pi\eta\theta}{l}.\left[\frac{x^4}{4}\right]_0^r = \frac{2\pi\eta\theta}{l}.\frac{r^4}{4} = \frac{\pi\eta\theta}{2l}.r^4$

We define torsional rigidity as twisting couple per unit angle of twist of a wire. This can be obtained by putting $\theta = 1\ (radian)$.

Torsional rigidity = $\frac{\pi\eta}{2l}.r^4$

STRAIN ENERGY IN THE TORSION

We know that strain energy per unit volume is equal to half the product of stress and strain.

$$W' = \frac{1}{2}F.\varphi$$

But $\eta = \frac{F}{\varphi}$, hence

$$W' = \frac{1}{2}\frac{F^2}{η}$$

If we consider an element of cylinder with radii x and x+dx, the stress will be constant at all points on this elementary cylinder and its value will be $x\, F_m/r$. Thus at x= r, stress will beF_m. This is the maximum value of stress in the wire which will be at the surface of the wire.

Since the volume of cylindrical element is $2\pi.x.l.dx$ hence energy of cylindrical element is

$$dE = \frac{1}{2}\left(\frac{x}{r}\right)^2 (F_m^2/η)\,(2\,\pi x l dx) = \frac{\pi l F_m^2.x^3 dx}{η r^2}$$

Integrating we get

$$E = \frac{\pi l F_m^2.}{η r^2}\int_0^r x^3 dx = \frac{\pi l F_m^2.}{4η r^2} r^4 = \frac{\pi r^2 l F_m^2.}{4η}$$

If φ_m is the maximum shear strain corresponding to the maximum value of stress F_m we have $F_m = η\varphi_m = η\frac{r\theta_m}{l}$. Here θ_m is angle of twist for maximum value of couple.

This gives me

$$E = \frac{\pi r^2 l.}{4n}\frac{η^2 r^2 \theta_m}{l^2} = \frac{1}{2}\,\frac{πη\theta_m}{2l}.r^4\theta_m$$

This means that $E = \frac{1}{2}C_m\theta_m$. Because $\frac{πn\theta_m}{2l} = C_m$

The strain energy is half the value it would have if all the elements of the wire were subjected to same maximum strain.

SELF-TEST 24

(1) The values of Y, η and K (Young's modulus, coefficient of rigidity and Bulk Modulus) for brass is 90×10^9, 35×10^9 and 75×10^9 respectively. The Poisson's ratio will be

(A) 1.0

(B) 0.5

(C) 0.3

(D) 0.1

(2) Find Y using the formula for conversion of elastic constants for Steel whose Bulk modulus and Coefficient of rigidity are 130×10^9 and 80×10^9 . The value obtained is closest to

(A) 50×10^9

(B) 100×10^9

(C) 150×10^9

(D) 200×10^9

Short Answer Questions 02

(1) Derive the relationship between the three moduli of elasticity.

(2) Explain the concept of Poisson's ratio.

(3) What is meant by primary and secondary strain?

(4) Derive the relation between modulus of elasticity and Poisson's ratio.

(5) Show that work done (per unit volume of material) in tensile stain is equal to half the product of strain and stress.

(6) Show that work done (per unit volume of material) in volume stain is equal to half the product of strain and stress.

(7) Show that work done (per unit volume of material) in shearing stain is equal to half the product of strain and stress.

(8) Show that work done (per unit volume of material) in torsion stain is equal to half the product of strain and stress.

01-03: Determination of Rigidity modulus by static torsion – Torsional Pendulum

A torsional pendulum consists of a rigid body suspended by a light wire or spring (Figure 01-09). When the body is twisted through some small maximum angle (Θ) and released from rest, the body oscillates between ($\theta = + \Theta$) and ($\theta = - \Theta$) .

The restoring torque is supplied by the shearing of the string or wire.

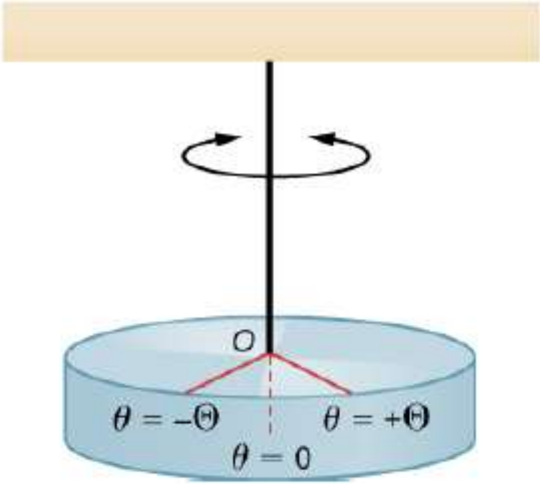

Figure 01-09: A torsional pendulum consists of a rigid body suspended by a string or wire. The rigid body oscillates between $\theta = +\Theta$ and $\theta = -\Theta$.

The restoring torque can be modeled as being proportional to the angle:

$\tau = -\kappa\theta$.

The variable kappa (κ) is known as the torsion constant of the wire or string. The minus sign shows that the restoring torque acts in the opposite direction to increasing angular displacement. The net torque is equal to the moment of inertia times the angular acceleration:

$$I\frac{d^2\theta}{dt^2} = -\kappa\theta;$$

$$\frac{d^2\theta}{dt^2} = -\frac{\kappa}{I}\theta.$$

This equation says that the second time derivative of the position (in this case, the angle) equals a negative constant times the position.

$$\frac{d^2x}{dt^2} = -\frac{k}{m}x,$$

This looks very similar to the equation of motion for the SHM where the period was found to be

$$T = 2\pi\sqrt{\frac{m}{k}}.$$

Therefore, the period of the torsional pendulum can be found using

$$\boxed{T = 2\pi\sqrt{\frac{I}{\kappa}}.}$$

(01-13)

TORSIONAL PENDULUM AND ELASTICITY

Let θ be the angle by which the wire was twisted. Then the resoting couple set up is equal to Cθ. Where C is the twisting couple per unit radian $(=\frac{\pi\eta}{2l}.r^4)$ as seen in earlier derivations.

We can write

$$I.\frac{d\omega}{dt} = -C\theta$$

$$\frac{d\omega}{dt} = -\frac{C}{I}\theta$$

The angular acceleration is proportional to angular displacement hence it satisfies condition of SHM, giving us the time period

$$T = 2\pi\sqrt{\frac{I}{C}}$$

(01-14)

Here C is the twisting couple per unit radian and I is the Moment of inertia. We can find the value of the coefficient of rigidity n using the relation between T,C and η.

$$C = \frac{4\pi^2 I}{T^2}$$

$$\frac{\eta\pi r^4}{2l} = \frac{4\pi^2 I}{T^2}$$

$$\eta = \frac{4\pi I l}{T r^4}$$

Here the challenge is to find Moment of Inertial accurately. For which Maxwell's needle method is used.

SOLVED PROBLEMS

MEASURING THE TORSION CONSTANT OF A STRING

A rod has a length of l = 0.30 m and a mass of 4.00 kg. A string is attached to the CM of the rod and the system is hung from the ceiling.

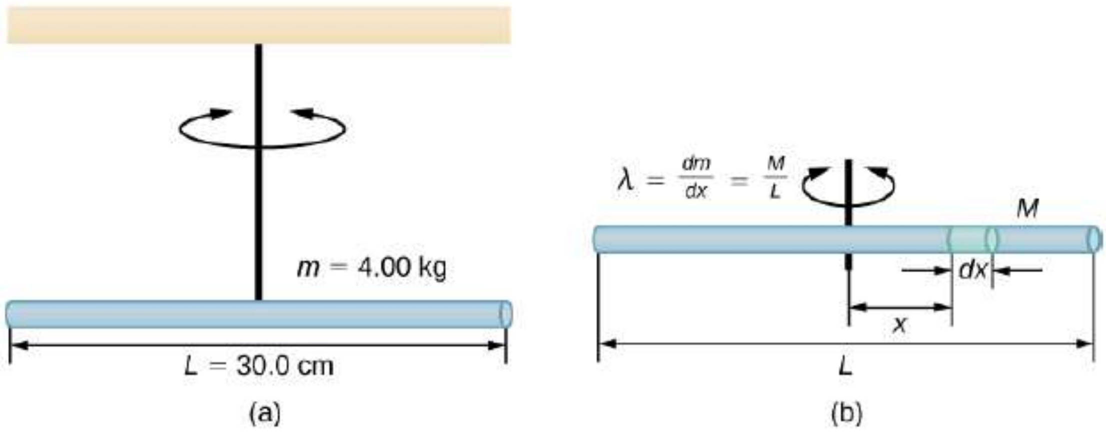

Fig 01-10: Example problem

The rod is displaced 10 degrees from the equilibrium position and released from rest. The rod oscillates with a period of 0.5 s. What is the torsion constant κ?

SOLUTION

1. Find the moment of inertia for the CM:

$$I_{CM} = \int x^2 dm = \int_{-L/2}^{+L/2} x^2 \lambda dx = \lambda\left[\frac{x^3}{3}\right]_{-L/2}^{+L/2} = \lambda\frac{2L^3}{24} = \left(\frac{M}{L}\right)\frac{2L^3}{24} = \frac{1}{12}ML^2.$$

2. Calculate the torsion constant using the equation for the period:

$$\begin{aligned} T &= 2\pi\sqrt{\frac{I}{\kappa}}; \\ \kappa &= I\left(\frac{2\pi}{T}\right)^2 = \left(\frac{1}{12}ML^2\right)\left(\frac{2\pi}{T}\right)^2; \\ &= \left(\frac{1}{12}(4.00\text{ kg})(0.30\text{ m})^2\right)\left(\frac{2\pi}{0.50\text{ s}}\right)^2 = 4.73\text{ N}\cdot\text{m}. \end{aligned}$$

SELF-TEST 25

(1) A rod has a length of l = 0.30 m and a mass of 4.00 kg. A string is attached to the CM of the rod and the system is hung from the ceiling. The rod is displaced 10 degrees from the equilibrium position and released from rest. The rod oscillates with a period of 0.5 s. The torsion constant (κ) found using the equation for period is closest to

(A) 2.7 N m

(B) 3.2 N m

(C) 4.7 N m

(D) 6.1 N m

(2) The coefficient of rigidity for a material under oscillation of torsion pendulum is given by

(A) $\eta = 4 \pi I l/Tr^4$

(B) $\eta = 4 \pi I l/Tr^2$

(C) $\eta = 4 \pi I l^2/Tr^4$

(D) $\eta = 4 \pi^2 I l/Tr^3$

Short Answer Questions 03

(1) Show that the torsional pendulum executes SHM. Is the motion simple harmonic only for small value of angle of rotation?

(2) How can we find the constants of elasticity using a torsion pendulum?

01-04: Searle's Method

Searle's apparatus consists of two metal frames F1 and F2. Each frame has a torsion head at the upper side and a hook at the lower side. These frames are suspended from two wires AB and CD of same material, length and cross-section. The upper ends of the wires are screwed tightly in two torsion heads fixed in the same rigid support. A spirit level rests horizontally with one end hinged in the frame F2. The other end of the spirit level rests on the tip of a spherometer screw, fitted in the frame F1. The spherometer screw can be rotated up and down along a vertical pitch scale marked in millimeters. The two frames are kept together by cross bars E1 and E2.

Two wires of the same material, length and diameter have their ends tightened in torsion screws A,B, C and D as shown in Figure 01-11. Wire AB (for which the Y is to be calculated) is called the experimental wire and CD is called the auxiliary wire (because it helps in determination of Y, though we don't wish to find Y for CD).

Suspend a know mass (e.g.1 kg) of dead load from hook of frames F1 and F2. The weight hanger at F1 is loaded and unloaded 3 or 4 times, so that the experimental wire AB comes under elastic mode. Now, each wire has been loaded equally with 1 kg. The pitch and the least count of the spherometer are determined. The central screw is adjusted in such a way that the air bubble in the spirit level is exactly at the centre. The head scale reading of the spherometer is noted for zero weight in the weight hanger attached to the frame F1. Another mass (say half kg) is now added to the weight hanger attached to the frame F1. The air bubble moves away from the centre. The spherometer screw is adjusted so that the air bubble comes back to the centre. The spherometer reading is noted.

The load is increased in steps of half kg (maximum load should be less than the breaking stress) and the corresponding spherometer reading is noted.

The same procedure is repeated for unloading the weights in steps of half kg. From these observations the extension, l for a load M can be determined

The arrangement is shown in following figure:

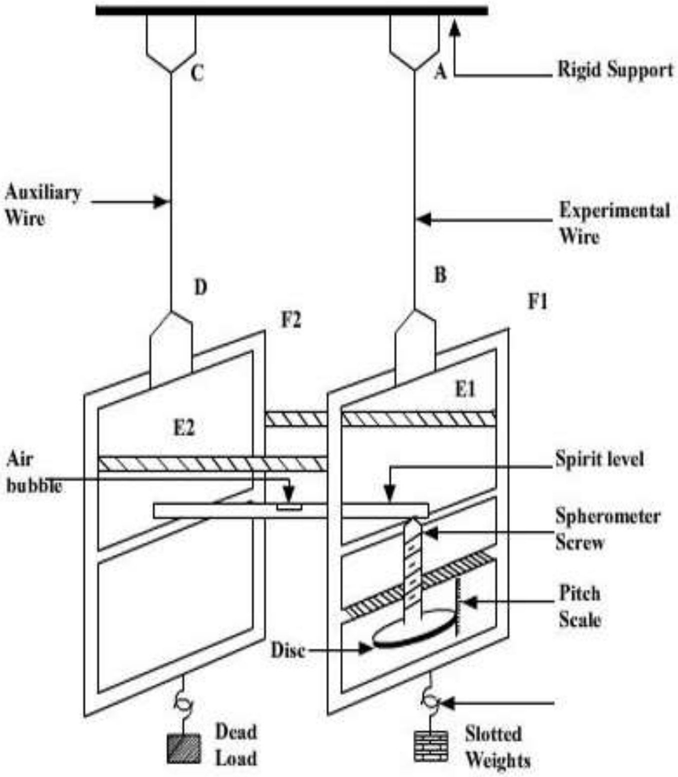

Fig 1-11: Searle's apparatus to measure Young's modulus

If a wire of length L and radius r be loaded by a weight Mg and if l is the extension produced,

Then, normal stress = $\frac{Mg}{\pi r^2}$

And Longitudinal strain = $\frac{l}{L}$

Hence, Young's modulus

$$Y = \frac{Normal\ Stress}{Longitudinal\ strain}$$

$$Y = \frac{Mg / \pi r^2}{l / L}$$

$$Y = \frac{MgL}{\pi r^2 l}$$

Where, L is length of the wire, l is extension for a load M, r is Radius of the wire, g is the acceleration due to gravity, M is the mass added in the hanger,

Young's modulus can be calculated using the above equation for various values of extensions corresponding to different loads M.

You will do an experiment (as described in Lab 06 in this book) in the laboratory part of the course.

SOLVED PROBLEMS

Example 01: During Searle's experiment, zero of the Vernier scale lies between 3.20×10^{-2} m and 3.25×10^{-2} m of the main scale. The 20th division of the Vernier scale exactly coincides with one of the main scale divisions. When an additional load of 2 kg is applied to the wire, the zero of the Vernier scale still lies between 3.20×10^{-2} m and 3.25×10^{-2} m of the main scale but now 45th division of Vernier scale coincides with one of the main scale divisions. The length of the thin metallic wire is 2 m and its cross-sectional area is 8×10^{-7} m^2. The least count of the Vernier scale is $1.0\times10-5$ m. Find the maximum percentage error in the Young's modulus?

Solution: The difference between two measurements by Vernier scale gives elongation of the wire caused by additional load of 2 kg. In first measurement, main scale reading is MSR=3.20×10^{-2}m and Vernier scale reading is VSR=20. The least count of Vernier scale is LC=1×10^{-5}m. Thus, first measurement by Vernier scale is,

L1=MSR+VSR×LC=$3.20\times10^{-2} + 20(1\times10^{-5}) = 3.220\times10^{-2}$ m.

In second measurement, MSR=3.20×10^{-2} m and VSR=45. Thus, second measurement by Vernier scale is,

L2 = $3.20\times10^{-2} + 45(1\times10^{-5}) = 3.245\times10^{-2}$ m.

The elongation of the wire due to force $F = 2kg$ is,

$l = L2 - L1 = 0.025\times10^{-2}$m.

The maximum error in measurement of l is $\Delta l = LC = 1\times10^{-5}$ m. Young's modulus is given by

$Y = \frac{FL}{lA}$.

The maximum percentage error in measurement of Y is,

$$\frac{\Delta Y}{Y}\times 100=\frac{\Delta l}{l}\times 100=\frac{1\times 10^{-5}}{0.025\times 10^{-2}}\times 100=4\%.$$

EXAMPLE 02:

In a Searle's experiment, the diameter of the wire as measured by a screw gauge of least count 0.001 cm is 0.050 cm. The length, measured by a scale of least count of 0.1 cm, is 110.0 cm. When a weight of 50 N is suspended from the wire, the extension is measured to be 0.125 cm by a micrometer of least count 0.001 cm. Find the maximum error in the measurement of Young's modulus of the material of wire from these data.

Solution: The Young's modulus is given by,

$$Y=\frac{4FL}{\pi d^2 l}=\frac{4(50)(110.0)}{3.14(0.050)^2(0.125)}=2.24\times 10^7\ \mathrm{N/cm^2}=2.24\times 10^{11}\ N/m^2.$$

Differentiate the expression for Y and simplify to get,

$$\frac{\Delta Y}{Y}=\frac{\Delta L}{L}+2\frac{\Delta d}{d}+\frac{\Delta l}{l}=\frac{0.1}{110.0}+\frac{2\times 0.001}{0.050}+\frac{0.001}{0.125}=0.049.$$

Thus, $\Delta Y=1.09\times 10^{10}\ \mathrm{N/m^2}$.

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Elasticity https://www.youtube.com/watch?v=HALbtyDUjp0

Static Equilibrium - Tension, Torque, Lever, Beam, & Ladder Problem – Physics https://www.youtube.com/watch?v=qGvFAl5CK_c

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COURSE COMPANION WEBSITE

Visit Here for Course Companion website for this course:

UNIT 04-02: SPECIAL THEORY OF RELATIVITY

LEARNING OBJECTIVES

After successful completion of this unit, you will be able to

- State the postulates of special theory of relativity (STR).
- Explain the Michelson-Morley experiment which supported the postulate of constancy of the speed of light in vacuum
- Explain concepts of Galilean transformation and Lorentz transformation
- Derive the Lorentz transformation using postulates of STR
- Discuss the concept of time dilation and derive the expression for time dilation using Lorentz transformation
- Discuss the concept of simultaneity
- Discuss the concept of length contraction and derive the expression for length contraction using Lorentz transformation

INTRODUCTION

Have you ever looked up at the night sky and dreamed of travelling to other planets in faraway star systems? Would there be other life forms? What would other worlds look like? You might imagine that such an amazing trip would be possible if we could just travel fast enough, but you will read in this chapter why this is not true. In 1905 Albert Einstein developed the theory of special relativity. This theory explains the limit on an object's speed and describes the consequences.

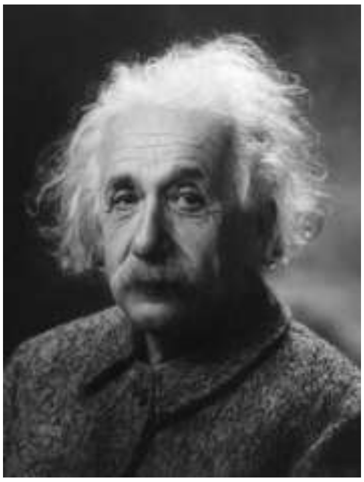

Figure 02-01: Many people think that Albert Einstein (1879–1955) was the greatest physicist of the 20th century. Not only did he develop modern relativity, thus

revolutionizing our concept of the universe, he also made fundamental contributions to the foundations of quantum mechanics. (credit: The Library of Congress)

Relativity: The word relativity might conjure an image of Einstein, but the idea did not begin with him. People have been exploring relativity for many centuries. Relativity is the study of how different observers measure the same event. Galileo and Newton developed the first correct version of classical relativity. Einstein developed the modern theory of relativity. Modern relativity is divided into two parts. Special relativity deals with observers who are moving at constant velocity. General relativity deals with observers who are undergoing acceleration. Einstein is famous because his theories of relativity made revolutionary predictions.

Most importantly, his theories have been verified to great precision in a vast range of experiments, altering forever our concept of space and time.

02-01: Postulates of Special Theory of Relativity.

Figure 02-02 Special relativity resembles trigonometry in that both are reliable because they are based on postulates that flow one from another in a logical way. (credit: Jon Oakley, Flickr)

Have you ever used the Pythagorean Theorem and got a wrong answer? You may probably not, unless you make a mistake in either algebra or arithmetic. Each time you perform the same calculation, you know that the answer will be the same.

Trigonometry is reliable because of the certainty that one part always flows from another in a logical way. Each part is based on a set of postulates, and you can always connect the parts by applying those postulates. Physics is the same way with exception that all parts must describe nature. If we are careful to choose the correct postulates, then our theory will follow and will be verified by experiment.

Einstein essentially did the theoretical aspect of this method of relativity. With two deceptively simple postulates and a careful consideration of how measurements are made, he produced the theory of special relativity.

Einstein's First Postulate

The first postulate upon which Einstein based the theory of special relativity relates to reference frames. All velocities are measured relative to some frame of reference. For example, a car's motion is measured relative to its starting point or the road it is moving over, a projectile's motion is measured relative to the surface it was launched from, and a planet's orbit is measured relative to the star it is orbiting around. The simplest frames of reference are those that are not accelerated and are not rotating.

Newton's first law, the law of inertia, holds exactly in such a frame.

Inertial Reference Frame

An inertial frame of reference is a reference frame in which a body at rest remains at rest and a body in motion moves at a constant speed in a straight line unless acted on by an outside force.

The laws of physics seem to be simplest in inertial frames. For example, when you are in a plane flying at a constant altitude and speed, physics seems to work exactly the same as if you were standing on the surface of the Earth. However, in a plane that is taking off, matters are somewhat more complicated. In these cases, the net force on an object, F, is not equal to the product of mass and acceleration, ma. Instead, F is equal to ma plus a fictitious force. This situation is not as simple as in an inertial frame. Not only are laws of physics simplest in inertial frames, but they should be the same in all inertial frames, since there is no preferred frame and no absolute motion. Einstein incorporated these ideas into his first postulate of special relativity.

First Postulate of Special Relativity

The laws of physics are the same and can be stated in their simplest form in all inertial frames of reference.

As with many fundamental statements, there is more to this postulate than meets the eye. The laws of physics include only those that satisfy this postulate. We shall find that the definitions of relativistic momentum and energy must be altered to fit. Another outcome of this postulate is the famous equation $E = mc^2$.

Einstein's Second Postulate

The second postulate upon which Einstein based his theory of special relativity deals with the speed of light. Late in the 19th century, the major tenets of classical physics were well established. Two of the most important were the laws of electricity and magnetism and Newton's laws. In particular, the laws of electricity and magnetism predict that light travels at $c = 3.00\times10^8$ m/s in a vacuum, but they do not specify the frame of reference in which light has this speed.

There was a contradiction between this prediction and Newton's laws, in which velocities add like simple vectors. If the latter were true, then two observers moving at different speeds would see light traveling at different speeds. Imagine what a light wave would look like to a person travelling along with it at a speed c. If such a motion were possible then the wave would be stationary relative to the observer. It would have electric and magnetic fields that varied in strength at various distances from the observer but were constant in time. This is not allowed by Maxwell's equations. So either Maxwell's equations are wrong, or an object with mass cannot travel at speed c. Einstein concluded that the latter is true. An object with mass cannot travel at speed c. This conclusion implies that light in a vacuum must always travel at speed c relative to any observer. Maxwell's equations are correct, and Newton's addition of velocities is not correct for light.

Investigations such as Young's double slit experiment in the early-1800s had convincingly demonstrated that light is a wave.

Many types of waves were known, and all travelled in some medium. Scientists therefore assumed that a medium carried light, even in a vacuum, and light travelled at a speed c relative to that medium. Starting in mid-1880, American physicist A.A. Michelson, later aided by E. W. Morley, made a series of direct measurements of the speed of light. The results of their measurements were startling.

MICHELSON-MORLEY EXPERIMENT

The Michelson-Morley experiment demonstrated that the speed of light in a vacuum is independent of the motion of the Earth about the Sun.

The eventual conclusion derived from this result is that light, unlike mechanical waves such as sound, does not need a medium to carry it. Furthermore, the Michelson-Morley results implied that the speed of light c is independent of the motion of the source relative to the observer. That is, everyone observes light to move at speed c regardless of how they move relative to the source or one another. For a number of years, many scientists tried unsuccessfully to explain these results and still retain the general applicability of Newton's laws.

It was not until 1905, when Einstein published his first paper on special relativity, that the currently accepted conclusion was reached. Based mostly on his analysis that the laws of electricity and magnetism would not allow another speed for light, and only slightly aware of the Michelson-Morley experiment, Einstein detailed his second postulate of special relativity.

SECOND POSTULATE OF SPECIAL RELATIVITY

The speed of light c is a constant, independent of the relative motion of the source.

Deceptively simple and counterintuitive, this and the first postulate leave all else open for change. Some fundamental concepts do change. Among the changes are the loss of agreement on the elapsed time for an event, the variation of distance with speed, and the realization that matter and energy can be converted into one another. You will read about these concepts in the following sections.

MISCONCEPTION ALERT: CONSTANCY OF THE SPEED OF LIGHT

The speed of light is a constant $c = 3.00 \times 10^8$ m/s in a vacuum. If you remember the effect of the index of refraction from the Law of Refraction, the speed of light is lower in matter.

GALILEAN TRANSFORMATION AND LORENTZ TRANSFORMATION

We have studied frame of references earlier in this course. We also had seen that if a frame S is moving in relation to another frame S' then the observation in frame S shows results different than that in S'. Suppose, for example, a bus and a train is moving on parallel paths. A person on the platform measures the speed of bus to be 60 kmph from left to right and measures the train as 100 kmph also from left to right. A person who is on train will observe the bus to be moving from right to left (moving backwards, just like the trees and stones on road are moving backwards) with a speed of 40 kmph. The person in bus will observe the train to be moving from left to right (in forward direction) with a speed of 40 kmph.

This is the simple example of relativity. If S' is moving with respect to S with a relative speed v. Let any point A be denoted by coordinates (x,y,z) in S and by (x',y',z') in S'. Similarly an event occurring at point A may be denoted by (x,y,z,t) and (x',y',z',t') in S and S' respectively. The four parameters like (x,y,z,t) which determine or describe an event is called space-time coordinates.

We have seen earlier that if v is in the positve X-axis,

$x' = x - vt$

$y' = y, \quad z' = z \quad \text{and } t' = t$

Obviously, if the frame S' is moving with velocity (v_x, v_y, v_z) along any direction with respect to S, the coordinates of point A in the two frames will be given by

$x' = x-v_x t$, $y' = y- v_y t$, $z' = z-v_z t$, $t' = t$

This is called Galilean transformation. It gives fairly good agreement with the experimental data when speed of frame is much less than speed of light in vacuum (c).

When the speed of frames of reference is comparable to the speed of light c, we have to use the transformation to be derived from the Einstein's postulates, which we discussed in the previous section. I will derive the transformation for such a case now.

Let S and S' be two inertial frames of reference such that S' is having a uniform velocity v relative to S. For convenience I am assuming that X-axes of the S and S' coincide.

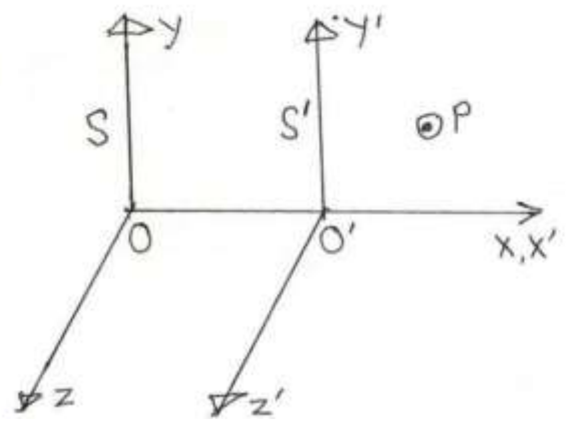

Fig 02-03: Frames S and S' are inertial frames of reference

Suppose that two observers O and O' observe event P from frames of reference S and S' respectively. Let us assume that event P is a light signal which is produced when at t'=0 and t=0 and when origins of S and S' coincide. Event P is observed as space-time coordinates (x,y,z,t) in S and (x',y'z',t') in S'.

The light pulse produced in event P will spread out as a growing sphere and the radius of the wavefront produced in this way will grow with speed c. Since (x,y,z,t) are the coordinates of event P in S, the equation for the spherical surface whose radius grows at speed c is given by:

$$x^2 + y^2 + z^2 = c^2t^2$$

(1)

Similarly the observer in S' is having coordinates of P as (x',y'z', t') and the equation of wavefront for that frame will be given by

$$x'^2 + y'^2 + z'^2 = c^2t'^2$$

(2)

Note that we did not have c’ due to the second postulate of Relativity (speed of light is constant in two inertial frames)

As velocity of S’ is in X-direction alone, hence by symmetry, we can conclude that

y = y’   and z = z’

(3)

Then from (1) and (2) we can have

$$x^2 - c^2t^2 = -(y^2 + z^2)$$

$$x'^2 - c^2t'^2 = -(y'^2 + z'^2)$$

Or,

$$x^2 - c^2t^2 = x'^2 - c^2t'^2$$

(4)

Now we know that for case v <<c, the transformation between x and x’ converges to Galilean transformation (x’ = x-vt). Also we wish the form of transformation as a linear transformation for simplicity. We will try the following trial solution for the transformation:

$$x' = \lambda(x - vt)$$

(5)

We may assume that (since motion is relative) S is moving relative to S’ with speed –v along the direction of +X axis, so that

$$x \quad = \lambda'(x' + vt')$$

(6)

We will put the value of x’ from (5) into (6)

$$x \quad = \lambda'[\lambda(x - vt) + vt']$$

$$\frac{x}{\lambda'} \quad = \lambda x - \lambda vt + vt'$$

Hence

$vt' = \frac{x}{\lambda'} + \lambda \quad v\,t - \lambda x = \lambda\left[\frac{x}{\lambda\lambda'} + vt - x\right]$

Thus,

$$t' = \lambda[t + \frac{x}{v\lambda\lambda'} - \frac{x}{v}]$$

Or,

$$t' = \lambda[t + \frac{x}{v}\left(\frac{1}{\lambda\lambda'} - 1\right)]$$

(7)

From equation (5) $x' = \lambda(x - vt)$, I can substitute this value of x' into equation (4)

I will get:

$$x^2 - c^2t^2 = [\lambda(x - vt)]^2 - c^2t'^2$$

Let me put the value of t' from (7) here:

$$x^2 - c^2t^2 = [\lambda(x - vt)]^2 - c^2\lambda^2\left[t + \frac{x}{v}\left(\frac{1}{\lambda\lambda'} - 1\right)\right]^2$$

$$x^2 - c^2t^2 - \lambda^2(x^2 - 2xvt + v^2t^2) + c^2\lambda^2\left[t^2 + 2\frac{xt}{v}\left(\frac{1}{\lambda\lambda'} - 1\right) + \frac{x^2}{v^2}\left(\frac{1}{\lambda\lambda'} - 1\right)^2\right] = 0$$

(8)

Since this equation is an identity, the coefficients of powers of x and t must vanish separately;

For collecting coefficient of x^2, I will highlight the terms in x^2 in the equation 8 (for your convenience and clarity)

$$x^2 - c^2t^2 - \lambda^2(x^2 - 2xvt + v^2t^2) + c^2\lambda^2\left[t^2 + 2\frac{xt}{v}\left(\frac{1}{\lambda\lambda'} - 1\right) + \frac{x^2}{v^2}\left(\frac{1}{\lambda\lambda'} - 1\right)^2\right] = 0$$

Collecting the terms in x^2:

$$1 - \lambda^2 + \frac{c^2\lambda^2}{v^2}\left(\frac{1}{\lambda\lambda'} - 1\right)^2 = 0$$

$$1 - \lambda^2 + \frac{c^2\lambda^2}{v^2}\left(\frac{1}{\lambda^2\lambda'^2} + 1 - \frac{2}{\lambda\lambda'}\right) = 0$$

$$1 - \lambda^2 + \frac{c^2}{v^2}[\lambda^2 - \frac{2\lambda}{\lambda'} + \frac{1}{\lambda'^2}] = 0$$

(9)

Let us now collect coefficient of xt and equate it to zero. As earlier, I am reproducing Eq (8) here and highlighting terms in xt for your convenience

$$x^2 - c^2t^2 - \lambda^2(x^2 - 2vxt + v^2t^2) + c^2\lambda^2\left[t^2 + 2\frac{xt}{v}\left(\frac{1}{\lambda\lambda'} - 1\right) + \frac{x^2}{v^2}\left(\frac{1}{\lambda\lambda'} - 1\right)^2\right] = 0$$

Hence I get:

$$2\,\lambda^2 v + \frac{c^2\lambda^2}{v}\left(\frac{1}{\lambda\lambda'} - 1\right) = 0$$

$$(v^2 - c^2)\lambda\lambda' + c^2 = 0$$

(10)

Lastly I need to collect the coefficient of t^2 and equate it to zero.

$-c^2 - \lambda^2 v^2 + c^2\lambda^2 = 0$

$(v^2 - c^2)\lambda^2 + c^2 = 0$

(11)

From 10 and (11), we have $\lambda = \lambda'$

Therefore equation (11) gives $\lambda = \frac{1}{\sqrt{\left(1-\frac{v^2}{c^2}\right)}}$

(12)

This value of λ can now be substituted in equation to obtain the transformations in the reference frames

$$x' = \frac{1}{\sqrt{\left(1-\frac{v^2}{c^2}\right)}}(x - vt)$$

(13)

From equation (7) and remembering that $\lambda = \lambda'$, we have

$$t' = \lambda\left[t - \frac{x}{v}\{1 - 1/\lambda^2\}\right]$$

Putting the value of λ from (13) we get;

$$t' = \frac{1}{\sqrt{\left(1-\frac{v^2}{c^2}\right)}}\left[t - \frac{x}{v}\left\{1 - \frac{c^2 - v^2}{c^2}\right\}\right]$$

$$t' = \frac{1}{\sqrt{\left(1-\frac{v^2}{c^2}\right)}}\left[t - \frac{x}{\cancel{v}}\left\{\frac{v^{\cancel{2}}}{c^2}\right\}\right]$$

$$t' = \frac{1}{\sqrt{\left(1-\frac{v^2}{c^2}\right)}}\left[t - vx/c^2\right]$$

(14)

Thus we have found how the coordinate of an arbitrary even P will be described in S' frame if they are described by (x,y,z,t) in frame S.

This transformation is called Lorentz Transformation equation.

$$x' = \frac{1}{\sqrt{\left(1 - \frac{v^2}{c^2}\right)}}(x - vt)$$

$$y' = y, \quad z' = z \ and$$

$$t' = \frac{t - vx/c^2}{\sqrt{\left(1 - \frac{v^2}{c^2}\right)}}$$

(15)

Sometimes we use notation $\beta = \frac{v}{c}$ $and\ \gamma = \frac{1}{\sqrt{\left(1 - \frac{v^2}{c^2}\right)}}$ as a short-cut to write cumbersome expressions.

I leave it to you to express Lorentz transformation in terms of β and γ.

Solved Problems

Problem 02-01

Show that $x^2 + y^2 + z^2 - c^2t^2$is invariant under Lorentz transformation

Solution

$$x'^2 + y'^2 + z'^2 - c^2t'^2 = \left[\frac{1}{\sqrt{\left(1 - \frac{v^2}{c^2}\right)}}(x - vt)\right]^2 + y^2 + z^2 - c^2\left[\frac{t - vx/c^2}{\sqrt{\left(1 - \frac{v^2}{c^2}\right)}}\right]^2$$

$$= \left(\frac{c^2}{c^2 - v^2}\right)[(x - vt)^2 - c^2(t - vx/c^2)^2] + y^2 + z^2$$

$$= \left(\frac{c^2}{c^2 - v^2}\right)\left[x^2 + v^2t^2 - 2xvt \quad - c^2\left(t^2 + \frac{x^2v^2}{c^4} \quad - \frac{vx}{c^2}\right) \quad \right] + y^2 + z^2$$

$$= \left(\frac{1}{c^2 - v^2}\right)[(c^2 - v^2)(x^2 - c^2t^2)] + y^2 + z^2$$

$$= [(x^2 - c^2t^2)] + y^2 + z^2 = x^2 + y^2 + z^2 - c^2t^2$$

Problem 02-02

Show that for low values of v the Lorentz transformation approaches Galilean transformation

Solution

Consider the Lorentz transformation

$$x' = \frac{1}{\sqrt{\left(1-\frac{v^2}{c^2}\right)}}(x - vt),$$

$$y' = y, \quad z' = z \text{ and}$$

$$t' = \frac{t - vx/c^2}{\sqrt{\left(1-\frac{v^2}{c^2}\right)}}$$

If v <<c, then $\frac{v^2}{c^2} \rightarrow 0$, and we will have x'=x-vt, y' = y, z'= z and t' = t.

SELF-TEST 26

(1) Special Theory of relativity is applicable for

(A) for frames rotating with respect to each other

(B) for frames accelerating with respect to each other

(C) for frames spinning with respect to a Lab frame

(D) none of the above

(2) According to the postulates of Special Theory of Relativity

(A) Inertial frames of reference are equivalent

(B) Speed of light in any medium is constant

(C) Nothing (including material and non-material objects like shadows) can move faster than speed of light in vacuum

(D) Non-inertial frames of references are equivalent

SHORT ANSWER QUESTIONS 01

(1) State the two postulates of the Special Theory of Relativity.

(2) What are the various evidences to support the speed of light postulate?

(3) What are the various arguments in support of the first postulate of relativity?

(4) Derive the Lorentz transformation for two inertial frames.

(5) Why do the Lotentz transformation does not work for non-inertial frames?

TIME DILATION.

Let us consider frame S' moving in positive x direction relative to frame S with a uniform speed v with the origins of the two frames coinciding at t = 0. Let a clock be set at x in frame S which emits signals at interval $\Delta t = t_2 - t_1$.

$$\Delta t = t_2 - t_1$$

(16)

The interval observed by a person in S' will be

$$\Delta t' = t'_2 - t'_1$$

(17)

We can find the values of $t'_2 \, and \, t'_1$ in terms of $t_2 and \, t_1$ using the Lorentz transformations.

$$t'_1 = \frac{t_1 - vx_1/c^2}{\sqrt{\left(1 - \frac{v^2}{c^2}\right)}}$$

$$t'_2 = \frac{t_2 - vx_2/c^2}{\sqrt{\left(1 - \frac{v^2}{c^2}\right)}}$$

Thus

$$\Delta t' = t'_2 - t'_1 = \left(\frac{t_2 - t_1}{\sqrt{\left(1 - \frac{v^2}{c^2}\right)}}\right)$$

$$\Delta t' = \frac{\Delta t}{\sqrt{\left(1 - \frac{v^2}{c^2}\right)}}$$

(18)

It means that the time interval appears to be dilated or elongated for the moving observer.

For example, suppose v is 99% the speed of light, v = 0.99 c. Hence v/c = 0.99.

The value of $\frac{v^2}{c^2} = 0.9801$, that makes $\left(1 - \frac{v^2}{c^2}\right) = 0.0199$. Thus the $\sqrt{\left(1 - \frac{v^2}{c^2}\right)}$ will have a value of 0.141 and $\frac{1}{\sqrt{\left(1-\frac{v^2}{c^2}\right)}} = 7.088$.

Hence if a signal with interval of 1 second is measured in the frame of reference where it is at rest, the same interval will be observed as with interval 7.088 seconds in the frame which is moving in with a speed of 99% speed of light.

This is because as v becomes comparable to c, the value of $\left(1 - \frac{v^2}{c^2}\right)$ will approach a small number, which will make interval $\Delta t'$ larger than t .

Now let us discuss this idea in greater details.

Simultaneity And Time Dilation

Figure 02-04 Elapsed time for a foot race is the same for all observers, but at relativistic speeds, elapsed time depends on the relative motion of the observer and the event that is observed. (credit: Jason Edward Scott Bain, Flickr)

Do time intervals depend on who observes them? Intuitively, we expect the time for a process, such as the elapsed time for a foot race, to be the same for all observers. Our experience has been that disagreements over elapsed time have to do with the accuracy of measuring time. When we carefully consider just how time is measured, however, we will find that elapsed time depends on the relative motion of an observer with respect to the process being measured.

Simultaneity

Consider how we measure elapsed time. If we use a stopwatch, for example, how do we know when to start and stop the watch?

One method is to use the arrival of light from the event, such as observing a light turning green to start a drag race. The timing will be more accurate if some sort of electronic detection is used, avoiding human reaction times and other complications.

Now suppose we use this method to measure the time interval between two flashes of light produced by flash lamps. (See Figure 02-05) Two flash lamps with observer A midway between them are on a rail car that moves to the right relative to observer B. Observer B arranges for the light flashes to be emitted just as A passes B, so that both A and B are equidistant from the lamps when the light is emitted. Observer B measures the time interval between the arrivals of the light flashes. According to postulate 2, the speed of light is not affected by the motion of the lamps relative to B. Therefore, light travels equal distances to him at equal speeds. Thus observer B measures the flashes to be simultaneous.

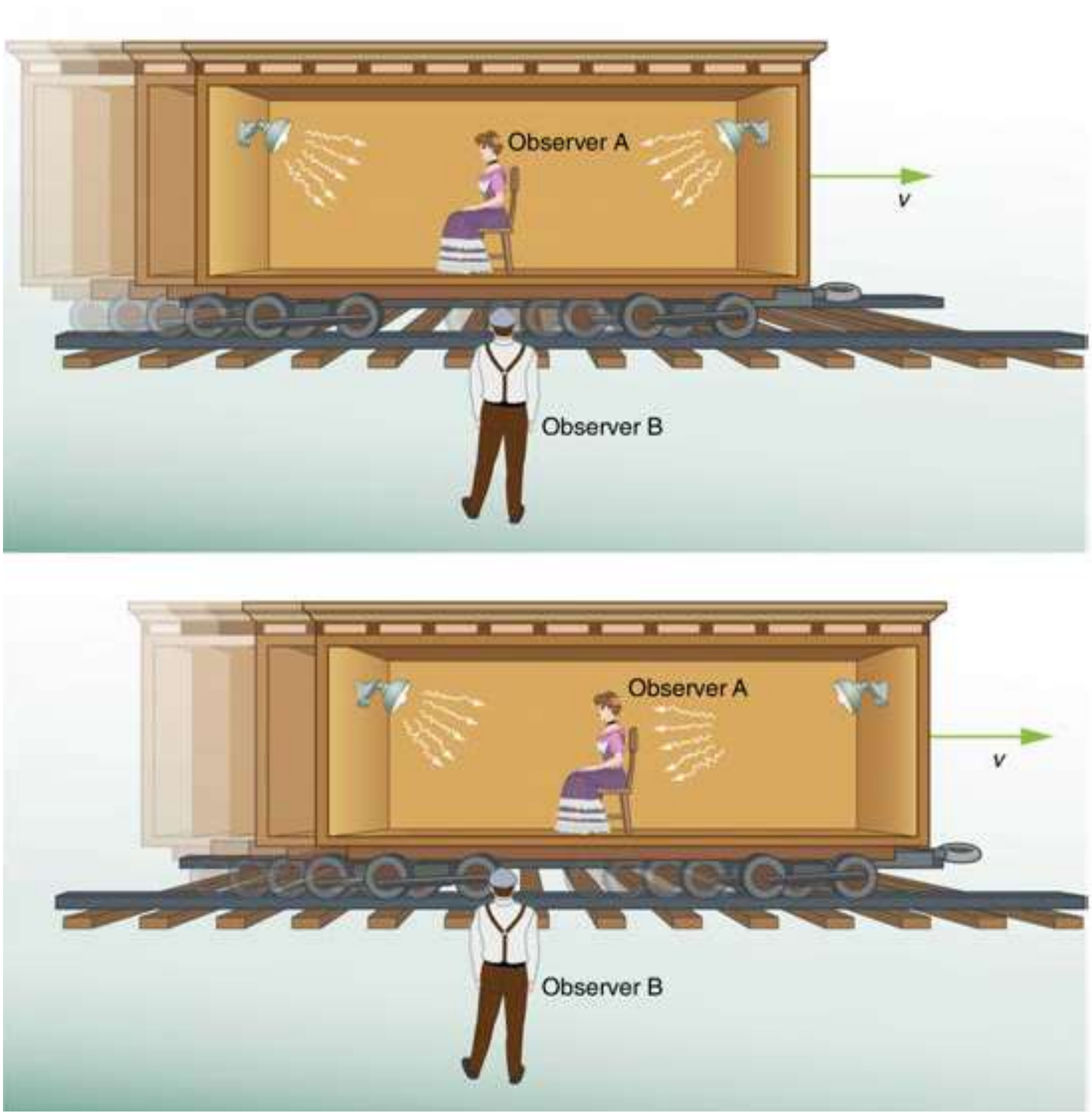

Figure 02-05 Observer B measures the elapsed time between the arrivals of light flashes

Now consider what observer B sees happen to observer A. Observer B perceives light from the right reaching observer A before light from the left, because she has moved towards that flash lamp, lessening the distance the light must travel and reducing the time it takes to get to her. Light travels at speed c relative to both observers, but observer B remains equidistant between the points where the flashes were emitted, while A gets closer to the emission point on the right. From observer B's point of view, then, there is a time interval between the arrival of the flashes to observer A. From observer B's point of view, then, there is a time interval between the arrival of the flashes to observer A. In observer A's frame of reference, the flashes occur at different times. Observer B measures the flashes to arrive simultaneously relative to him but not relative to A.

Now consider what observer A sees happening. She sees the light from the right arriving before light from the left. Since both lamps are the same distance from her in her reference frame, from her perspective, the right flash occurred before the left flash.

Here a relative velocity between observers affects whether two events are observed to be simultaneous. Simultaneity is not absolute.This illustrates the power of clear thinking. We might have guessed incorrectly that if light is emitted simultaneously, then two observers halfway between the sources would see the flashes simultaneously. But careful analysis shows this not the case.

Einstein was brilliant at this type of thought experiment (in German, "Gedankenexperiment"). He very carefully considered how an observation is made and disregarded what might seem obvious. The validity of thought experiments, of course, is determined by actual observation. The genius of Einstein is evidenced by the fact that experiments have repeatedly confirmed his theory of relativity.

In summary: Two events are defined to be simultaneous if an observer measures them as occurring at the same time (such as by receiving light from the events). Two events are not necessarily simultaneous to all observers.

TIME DILATION

The consideration of the measurement of elapsed time and simultaneity leads to an important relativistic effect.

Time dilation

Time dilation is the phenomenon of time passing slower for an observer who is moving relative to another observer.

Suppose, for example, an astronaut measure the time it takes for light to cross her ship, bounce off a mirror, and return. (See Figure 02-06). How does the elapsed time the astronaut measure compare with the elapsed time measured for the same event by a person on the Earth? Asking this question (another thought experiment) produces a profound result. We find that the elapsed time for a process depends on who is measuring it. In this case, the time measured by the astronaut is smaller than the time measured by the Earth-bound observer. The passage of time is different for the observers because the distance light travels in the astronaut's frame are smaller than in the Earth-bound frame. Light travels at the same speed in each frame, and so it will take longer to travel the greater distance in the Earth-bound frame.

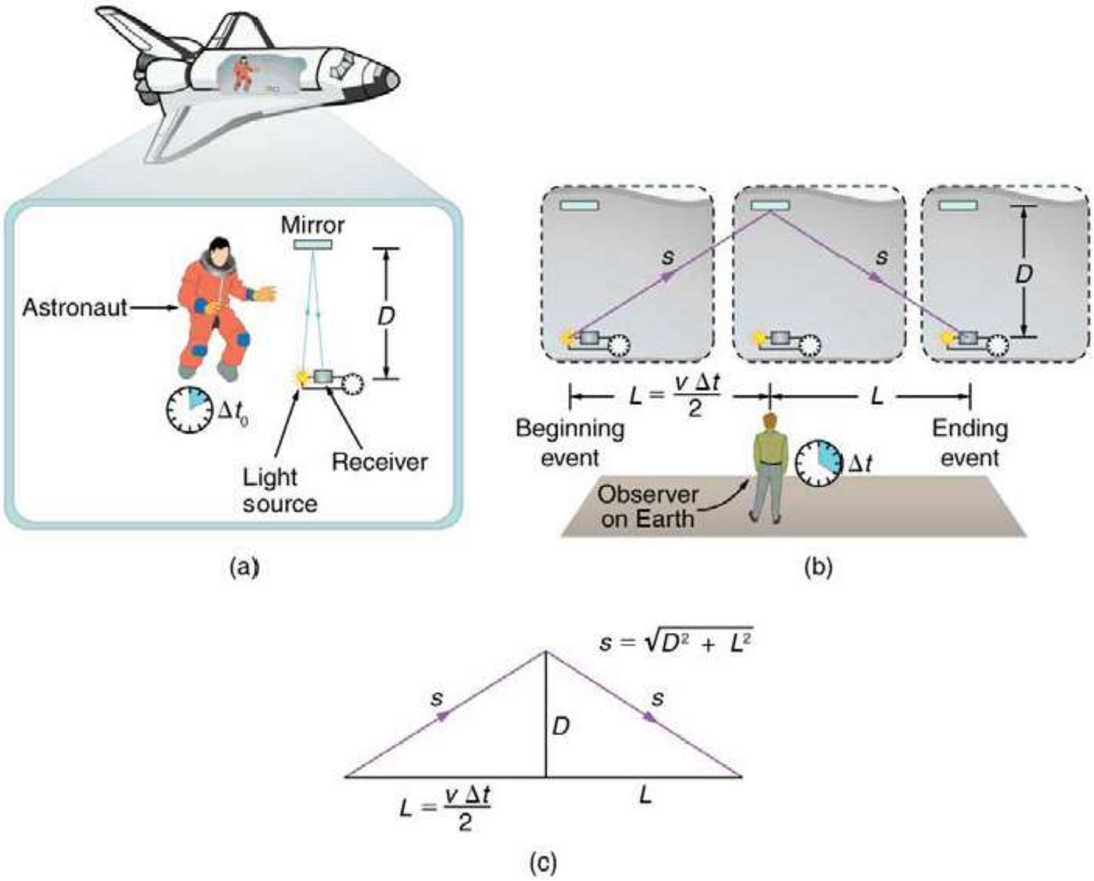

Figure 02.06 (a) An astronaut measures the time Δt_0 for light to cross her ship using an electronic timer. Light travels a distance 2D in the astronaut's frame. (b) A person on the Earth sees the light follow the longer path 2s and take a longer time Δt. (c) These triangles are used to find the relationship between the two distances 2D and 2s.

To quantitatively verify that time depends on the observer, consider the paths followed by light as seen by each observer. (See Figure 02-06) The astronaut sees the light travel straight across and back for a total distance of 2D, twice the width of her ship. The Earth-bound observer sees the light travel a total distance 2s. Since the ship is moving at speed v to the right relative to the Earth, light moving to the right hits the mirror in this frame. Light travels at a speed c in both frames, and because time is the distance divided by speed, the time measured by the astronaut is

$$\Delta t_0 = \frac{2D}{c}.$$

(19)

This time has a separate name to distinguish it from the time measured by the Earth-bound observer.

PROPER TIME

Proper time Δt_0 is the time measured by an observer at rest relative to the event being observed.

In the case of the astronaut observe the reflecting light, the astronaut measures proper time. The time measured by the earth-bound observer is

$$\Delta t = \frac{2s}{c}.$$

(20)

To find the relationship between Δt_0 and Δt, consider the triangles formed by D and s . (See Figure (c))

The third side of these similar triangles is L , the distance the astronaut moves as the light goes across her ship. In the frame of the earth-bound observer,

$$L = \frac{v\Delta t}{2}.$$

(21)

Using the Pythagorean Theorem, the distance s is found to be

$$s = \sqrt{D^2 + \left(\frac{v\Delta t}{2}\right)^2}.$$

(22)

Substituting s into the expression for the time interval Δt gives

$$\Delta t = \frac{2s}{c} = \frac{2\sqrt{D^2 + \left(\frac{v\Delta t}{2}\right)^2}}{c}.$$

(23)

We square this equation, which yields

$$(\Delta t)^2 = \frac{4\left(D^2 + \frac{v^2(\Delta t)^2}{4}\right)}{c^2} = \frac{4D^2}{c^2} + \frac{v^2}{c^2}(\Delta t)^2.$$

(24)

Note that if we square the first expression we had for Δt_0, we get $(\Delta t_0)^2 = \frac{4D^2}{c^2}$. This term appears in the preceding equation, giving us a means to relate the two time intervals. Thus,

$$(\Delta t)^2 = (\Delta t_0)^2 + \frac{v^2}{c^2}(\Delta t)^2.$$

(25)

Gathering terms, we solve for Δt :

$$(\Delta t)^2\left(1 - \frac{v^2}{c^2}\right) = (\Delta t_0)^2.$$

(26)

Thus

$$(\Delta t)^2 = \frac{(\Delta t_0)^2}{1 - \frac{v^2}{c^2}}.$$

(27)

Taking the square root yields an important relationship between elapsed times:

$$\Delta t = \frac{\Delta t_0}{\sqrt{1 - \frac{v^2}{c^2}}} = \gamma \Delta t_0,$$

(28)

Where

$$\gamma = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}.$$

This equation for Δt is truly remarkable. First, as contended, elapsed time is not the same for different observers moving relative to one another, even though both are in inertial frames. Proper time Δt_0 measured by an observer, like the astronaut moving with the apparatus, is smaller than time measured by other observers. Since those other observers measure a longer time Δt, the effect is called time dilation. The earth-bound observer sees time dilate (get longer) for a system moving relative to earth. Alternatively, according to the earth-bound observer, time slows in the moving frame, since less time passes there. All clocks moving relative to an observer, including biological clocks such as aging, are observed to run slow compared with a clock stationary relative to the observer.

Note that if the relative velocity is much less than the speed of light ($v << c$), then v_2 c_2 is extremely small, and the elapsed times Δt and Δt_0 are nearly equal. At low velocities, modern relativity approaches classical physics—our everyday experiences have very small relativistic effects.

The equation $\Delta t = \gamma\Delta t_0$ also implies that relative velocity cannot exceed the speed of light. As v approaches c, Δt approaches infinity. This would imply that time in the astronaut's frame stops at the speed of light. If v exceeded c, then we would be taking the square root of a negative number, producing an imaginary value for Δt.

There is considerable experimental evidence that the equation $\Delta t = \gamma\Delta t_0$ is correct. One example is found in cosmic ray particles that continuously rain down on the Earth from deep space. Some collisions of these particles with nuclei in the upper atmosphere result in short-lived particles called muons. The half-life (amount of time for half of a material to decay) of a muon is 1.52 μs when it is at rest relative to the observer who measures the half-life. This is the proper time Δt_0. Muons produced by cosmic ray particles have a range of velocities, with some moving near the speed of light. It has been found that the muon's halflife as measured by an earth-bound observer (Δt) varies with velocity exactly as predicted by the equation $\Delta t = \gamma\Delta t_0$. The faster the muon moves, the longer it lives. We on the earth see the muon's half-life time dilated—as viewed from our frame, the muon decays more slowly than it does when at rest relative to us.

SOLVED PROBLEMS

Problem

Suppose a cosmic ray colliding with a nucleus in the Earth's upper atmosphere produces a muon that has a velocity $v = 0.950c$. The muon then travels at constant velocity and lives 1.52 μs as measured in the muon's frame of reference.

(You can imagine this as the muon's internal clock.) How long does the muon live as measured by an earth-bound observer? (See Figure 02-07.)

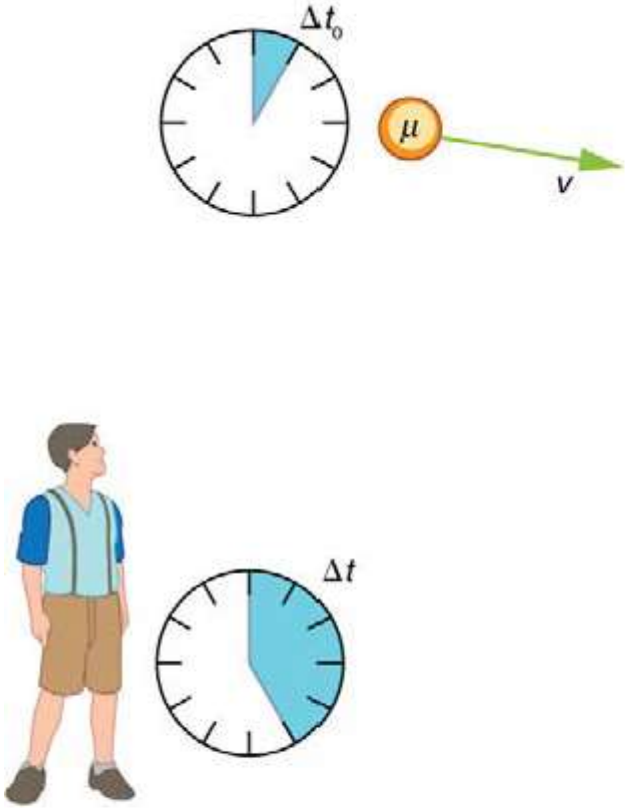

Fig 02-07: Muon's internal clock problem

Strategy

A clock moving with the system being measured observes the proper time, so the time we are given is Δt_0 = 1.52 μs. The earth-bound observer measures Δt as given by the equation $\Delta t = \gamma \Delta t_0$. Since we know the velocity, the calculation is straightforward.

Solution

1) Identify the known quantities. $v = 0.950c$, Δt_0 = 1.52 μs

2) Identify the unknown. Δt

3) Choose the appropriate equation.

Use,

$\Delta t = \gamma \Delta t_0$, where

$$\gamma = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}.$$

.

4) Plug the known into the equation.

First find γ .

$$\begin{aligned}\gamma &= \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}\\ &= \frac{1}{\sqrt{1 - \frac{(0.950c)^2}{c^2}}}\\ &= \frac{1}{\sqrt{1 - (0.950)^2}}\\ &= 3.20.\end{aligned}$$

This OpenStax book is available for free at http://cnx.org/content/col11406/1.9

Use the calculated value of γ to determine Δt .

$$\begin{aligned}\Delta t &= \gamma \Delta t_0\\ &= (3.20)(1.52\ \mu s)\\ &= 4.87\ \mu s\end{aligned}$$

Discussion

One implication of this example is that since γ = 3.20 at 95.0% of the speed of light (v = 0.950c), the relativistic effects are significant. The two time intervals differ by this factor of 3.20, where classically they would be the same. Something moving at 0.950c is said to be highly relativistic.

Another implication of the preceding example is that everything an astronaut does when moving at 95.0% of the speed of light relative to the Earth takes 3.20 times longer when observed from the Earth. Does the astronaut sense this? Yes, only if she looks outside her spaceship. All methods of measuring time in her frame will be affected by the same factor of 3.20. This includes her wristwatch, heart rate, cell metabolism rate, nerve impulse rate, and so on. She will have no way of telling, since all of her clocks will agree with one another because their relative velocities are zero. Motion is relative, not absolute. But what if she does look out the window?

Real-World Connections

It may seem that special relativity has little effect on your life, but it is probably more important than you realize. One of the most common effects is through the Global Positioning System (GPS). Emergency vehicles, package delivery services, electronic maps, and communications devices are just a few of the common uses of GPS, and the GPS system could not work without taking into account relativistic effects. GPS satellites rely on precise time measurements to communicate. The signals travel at relativistic speeds. Without corrections for time dilation, the satellites could not communicate, and the GPS system would fail within minutes.

Problem 02-00

A particle travels at 1.90×108 m/s and lives 2.10×10−8 s when at rest relative to an observer. How long does the particle live as viewed in the laboratory?

Solution

$$\Delta t = \frac{\Delta_t}{\sqrt{1-\frac{v^2}{c^2}}} = \frac{2.10\times10^{-8}\ \text{s}}{\sqrt{1-\frac{(1.90\times10^{8}\ \text{m/s})^2}{(3.00\times10^{8}\ \text{m/s})^2}}} = 2.71\times10^{-8}\ \text{s}$$

The Twin Paradox

An intriguing consequence of time dilation is that a space traveler moving at a high velocity relative to the Earth would age less than her Earth-bound twin. Imagine the astronaut moving at such a velocity that $\gamma = 30.0$, as in Figure 02-08. A trip that takes 2.00 years in her frame would take 60.0 years in her Earth-bound twin's frame. Suppose the astronaut travelled 1.00 year to another star system. She briefly explored the area, and then travelled 1.00 year back. If the astronaut was 40 years old when she left, she would be 42 upon her return. Everything on the Earth, however, would have aged 60.0 years. Her twin, if still alive, would be 100 years old.

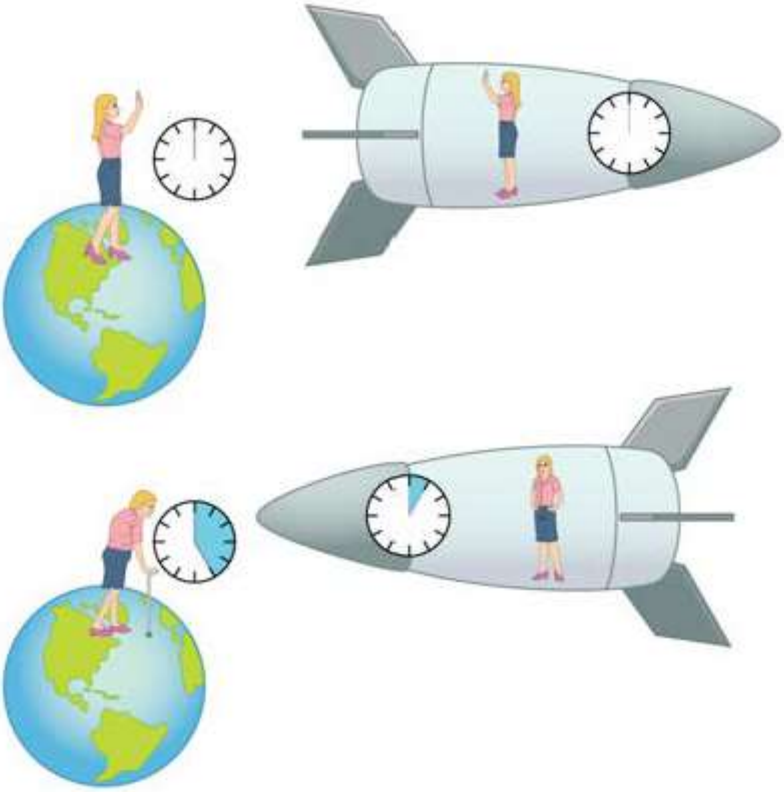

Fig 02-08: The twin paradox asks why the traveling twin ages less than the earth-bound twin. That is the prediction we obtain if we consider the earth-bound twin's frame. In the astronaut's frame, however, the Earth is moving and time runs slower there. Who is correct?

The situation would seem different to the astronaut. Because motion is relative, the spaceship would seem to be stationary and the earth would appear to move. (This is the sensation you have when flying in a jet.) If the astronaut looks out the window of the spaceship, she will see time slow down on the earth by a factor of $\gamma = 30.0$. To her, the earth-bound sister will have aged only 2/30 (1/15) of a year, while she aged 2.00 years. The two sisters cannot both be correct! Such presentations of situations where we get ambiguous or confusing conclusion are called paradoxes. This specific paradox is called twin paradox.

As with all paradoxes, the premise is faulty and leads to contradictory conclusions. In fact, the astronaut's motion is significantly different from that of the earth-bound twin. The astronaut accelerates to a high velocity and then decelerates to view the star system. To return to the earth, she again accelerates and decelerates. The earth-bound twin does not experience these accelerations. So the situation is not symmetric, and it is not correct to claim that the astronaut will observe the same effects as her earth-bound twin. If you use special relativity to examine the twin paradox, you must keep in mind that the theory is expressly based on inertial frames, which by definition are not accelerated or rotating. Einstein developed general relativity to deal with accelerated frames and with gravity, a prime source of acceleration. You can also use general relativity to address the twin paradox and, according to general relativity, the astronaut will age less.

In 1971, American physicists Joseph Hafele and Richard Keating verified time dilation at low relative velocities by flying extremely accurate atomic clocks around the Earth on commercial aircraft. They measured elapsed time to an accuracy of a few nanoseconds and compared it with the time measured by clocks left behind. Hafele and Keating's results were within experimental uncertainties of the predictions of relativity. Both special and general relativity had to be taken into account, since gravity and accelerations were involved as well as relative motion.

02-02: LENGTH CONTRACTION.

Consider two coordinate systems S and S' with S' moving with a relative velocity v wrt S in +X-axis. Let a rod be at rest in S' positioned along X-axis. Let L_0 be the length of rod in a frame (like S') where it is at rest. Let the x-coordinates of the two ends of the rod be x_1 and x_2 in S'.

$$L_0 = (x_2 - x_1)$$

(30)

If L is the length of the rod in frame S and x-coordinates of ends of the rod are x_1' and x_2', we can use the Lorentz transformation to find these coordinates in terms of x_1 and x_2.

$$x_2' = \frac{1}{\sqrt{\left(1-\frac{v^2}{c^2}\right)}}(x_2 - vt)$$

$$x_1' = \frac{1}{\sqrt{\left(1-\frac{v^2}{c^2}\right)}}(x_1 - vt)$$

(31)

Hence the length (L) of rod as measured in S:

$$L = (x'_2 - x'_1)$$

$$L = \frac{1}{\sqrt{\left(1-\frac{v^2}{c^2}\right)}}(x_2 - x_1)$$

(31)

$$L = \frac{L_0}{\sqrt{\left(1 - \frac{v^2}{c^2}\right)}}$$

$$L = L_0\sqrt{\left(1 - \frac{v^2}{c^2}\right)}$$

(32)

Thus for the moving observer the length of an object appears to be contracted or shortened.

For example, suppose v is 99% the speed of light, v = 0.99 c. Hence v/c = 0.99.

The value of $\frac{v^2}{c^2} = 0.9801$, that makes $\left(1 - \frac{v^2}{c^2}\right) = 0.0199$. Thus the $\sqrt{\left(1 - \frac{v^2}{c^2}\right)}$ will have a value of 0.141.

Hence if a rod of 100 cm (1metre) in the frame of reference where it is at rest, the same rod will be seen as with length 14.1cm in the frame which is moving in the direction of the length of the rod with a speed of 99% speed of light.

This is because as v becomes comparable to c, the value of $\left(1 - \frac{v^2}{c^2}\right)$ will approach a small number, which will make L smaller than L_0.

Let us now consider the concept of length contraction from a different perspective.

Fig 02-09: People might describe distances differently, but at relativistic speeds, the distances really are different. (credit: Corey Leopold, Flickr)

Have you ever driven on a road that seems like it goes on forever? If you look ahead, you might say you have about 10 km left to go. Another traveler might say the road ahead looks like it's about 15 km long. If you both measured the road, however, you would agree. Traveling at everyday speeds, the distance you both measure would be the same. You will read in this section, however, that this is not true at relativistic speeds. Close to the speed of light, distances measured are not the same when measured by different observers.

Proper Length

One thing all observers agree upon is relative speed. Even though clocks measure different elapsed times for the same process, they still agree that relative speed, which is distance divided by elapsed time, is the same. This implies that distance, too, depends on the observer's relative motion. If two observers see different times, then they must also see different distances for relative speed to be the same to each of them.

The muon discussed in Problem discussed earlier illustrates this concept. To an observer on the earth, the muon travels at 0.950c for 7.05 μs from the time it is produced until it decays. Thus it travels a distance

$$L_0 = v\Delta t = (0.950)(3.00\times10^8 \text{ m/s})(7.05\times10^{-6} \text{ s}) = 2.01 \text{ km}$$

relative to the Earth. In the muon's frame of reference, its lifetime is only 2.20 μs . It has enough time to travel only

$$L = v\Delta t_0 = (0.950)(3.00\times10^8 \text{ m/s})(2.20\times10^{-6} \text{ s}) = 0.627 \text{ km.}$$

The distance between the same two events (production and decay of a muon) depends on who measures it and how they are moving relative to it.

Proper Length

Proper length L_0 is the distance between two points measured by an observer who is at rest relative to both of the points.

The earth-bound observer measures the proper length L_0, because the points at which the muon is produced and decays are stationary relative to the Earth. To the muon, the earth, air, and clouds are moving, and so the distance L it sees is not the proper length.

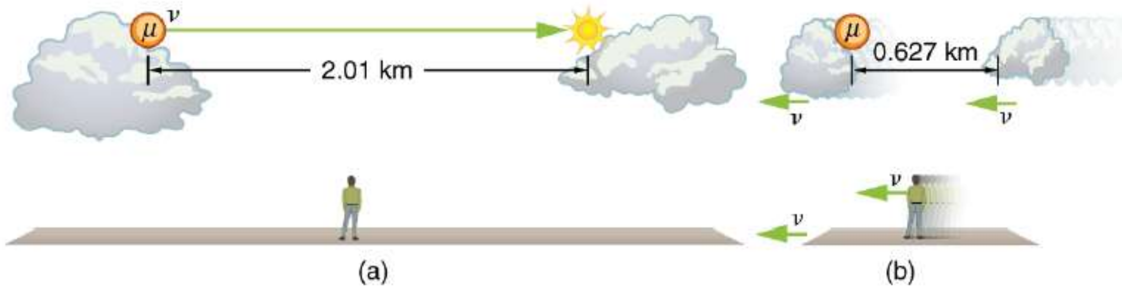

Fig 02-10: (a) The Earth-bound observer sees the muon travel 2.01 km between clouds. (b) The muon sees itself travel the same path, but only a distance of 0.627 km. The

Earth, air, and clouds are moving relative to the muon in its frame, and all appear to have smaller lengths along the direction of travel.

LENGTH CONTRACTION

To develop an equation relating distances measured by different observers, we note that the velocity relative to the earth-bound observer in our muon example is given by

$$v = \frac{L_0}{\Delta t}. \tag{33}$$

The time relative to the earth-bound observer is Δt, since the object being timed is moving relative to this observer. The velocity relative to the moving observer is given by

$$v = \frac{L}{\Delta t_0}. \tag{34}$$

The moving observer travels with the muon and therefore observes the proper time Δt_0 . The two velocities are identical; thus,

$$\frac{L_0}{\Delta t} = \frac{L}{\Delta t_0}. \tag{35}$$

We know that $\Delta t = \gamma \Delta t_0$. Substituting this equation into the relationship above gives

$$L = \frac{L_0}{\gamma}. \tag{36}$$

Substituting for γ gives an equation relating the distances measured by different observers.

We had derived this relation between L and L_0 using Lorentz transformation.

Length Contraction

Length contraction L is the shortening of the measured length of an object moving relative to the observer's frame.

$$L = L_0\sqrt{1 - \frac{v^2}{c^2}}.$$

If we measure the length of anything moving relative to our frame, we find its length L to be smaller than the proper length L_0 that would be measured if the object were stationary. For example, in muon's reference frame, the distance between the points where it was produced and where it decayed is shorter. Those points are fixed relative to earth but moving relative to muon. Clouds and other objects are also contracted along the direction of motion in the muon's reference frame.

Solved Problems

Problem 02-03

Suppose an astronaut, such as the twin discussed in Simultaneity and Time Dilation, travels so fast that $\gamma = 30.00$.

(a) She travels from the Earth to the nearest star system, Alpha Centauri, 4.300 light years (ly) away as measured by an earthbound observer. How far apart are the Earth and Alpha Centauri as measured by the astronaut?

(b) In terms of c, what is her velocity relative to the Earth? You may neglect the motion of the Earth relative to the Sun. (See Figure 02-09)

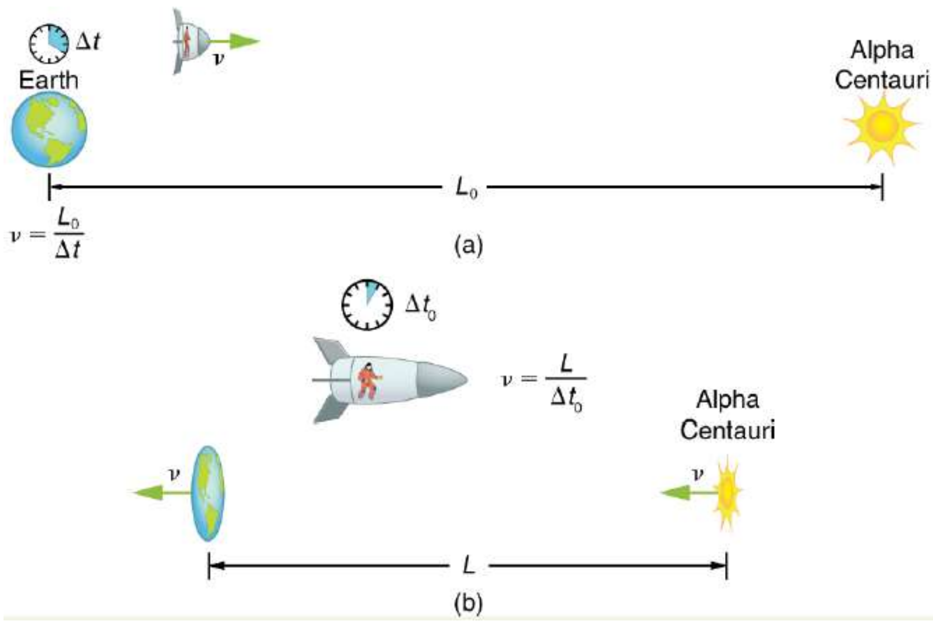

Fig 02-11: (a) The Earth-bound observer measures the proper distance between the Earth and the Alpha Centauri. (b) The astronaut observes a length contraction, since the Earth and the Alpha Centauri move relative to her ship. She can travel this shorter distance in a smaller time (her proper time) without exceeding the speed of light.

Strategy

First note that a light year (ly) is a convenient unit of distance on an astronomical scale—it is the distance light travels in a year. For part (a), note that the 4.300 ly distance between the Alpha Centauri and the earth is the proper distance L_0, because it is measured by an earth-bound observer to whom both stars are (approximately) stationary. To the astronaut, the earth and the Alpha Centauri are moving by at the same velocity, and so the distance between them is the contracted length L. In part (b), we are given γ, and so we can find v by rearranging the definition of γ to express v in terms of c.

Solution for (a)

1. Identify the known quantities. $L_0 = 4.300$ ly ; $\gamma = 30.00$

2. Identify the unknown. L

3. Choose the appropriate equation. $L = L_0/\gamma$

4. Rearrange the equation to solve for the unknown.

$$\begin{aligned} L &= \frac{L_0}{\gamma} \\ &= \frac{4.300 \text{ ly}}{30.00} \\ &= 0.1433 \text{ ly} \end{aligned}$$

Solution for (b)

1. Identify the known. $\gamma = 30.00$

2. Identify the unknown. v in terms of c

3. Choose the appropriate equation. $\gamma = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}$

4. Rearrange the equation to solve for the unknown.

$$\begin{aligned} \gamma &= \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} \\ 30.00 &= \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} \end{aligned}$$

Squaring both sides of the equation and rearranging terms gives

$$900.0 = \frac{1}{1 - \frac{v^2}{c^2}}$$

so that

$$1 - \frac{v^2}{c^2} = \frac{1}{900.0}$$

and

$$\frac{v^2}{c^2} = 1 - \frac{1}{900.0} = 0.99888....$$

Taking the square root, we find

$$\frac{v}{c} = 0.99944,$$

which is rearranged to produce a value for the velocity

v= 0.9994c.

Discussion

First, remember that you should not round off calculations until the final result is obtained, or you could get erroneous results. This is especially true for special relativity calculations, where the differences might only be revealed after several decimal places. The relativistic effect is large here (γ=30.00), and we see that v is approaching (not equaling) the speed of light. Since the distance as measured by the astronaut is so much smaller, the astronaut can travel it in much less time in her frame.

People could be sent very large distances (thousands or even millions of light years) and age only a few years on the way if they travelled at extremely high velocities. But, like emigrants of centuries past, they would leave the earth they know forever. Even if they returned, thousands to millions of years would have passed on the Earth, obliterating most of what now exists. There is also a more serious practical obstacle to travelling at such velocities; immensely greater energies than classical physics predicts would be needed to achieve such high velocities. This will be discussed in Relatavistic Energy.

Why don't we notice length contraction in everyday life? The distance to the grocery shop does not seem to depend on whether we are moving or not. Examining the equation

$$L = L_0\sqrt{1 - \frac{v^2}{c^2}},$$

we see that at low velocities (v <<c) the lengths are nearly equal, the classical expectation. But length contraction is real, if not commonly experienced. For example, a charged particle, like an electron, traveling at relativistic velocity has electric field lines that are compressed along the direction of motion as seen by a stationary observer. As the electron passes a detector, such as a coil of wire, its field interacts much more briefly, an effect observed at particle accelerators such as the 3 km long Stanford Linear Accelerator (SLAC). In fact, to an electron travelling down the beam pipe at SLAC, the accelerator and the Earth are all moving by and are length contracted. The relativistic effect is so great than the accelerator is only 0.5 m long to the electron. It is actually easier to get the electron beam down the pipe, since the beam does not have to be as precisely aimed to get down a short pipe as it would down one 3 km long.

This, again, is an experimental verification of the Special Theory of Relativity.

Problem 02-04

A particle is traveling through the earth's atmosphere at a speed of 0.750c. To an earth-bound observer, the distance it travels is 2.50 km. How far does the particle travel in the particle's frame of reference?

Solution

$$L=L_0\sqrt{1-\frac{v^2}{c^2}}=(2.50\text{ km})\sqrt{1-\frac{(0.750c)^2}{c^2}}=1.65\text{ km}$$

SELF-TEST 27

(1) A particle is traveling through the earth's atmosphere at a speed of 0.60c. To an earth-bound observer, the distance it travels is 1.00 km. How far does the particle travel in the particle's frame of reference?

(A) 1.6 km

(B) 0.8 km

(C) 0.6 km

(D) 1.0 km

(2) Length contraction occurs for a body traveling with respect to another frame of reference

(A) in all the directions

(B) only in the direction perpendicular to the direction of relative motion of the frames

(C) only in the direction of the motion of frames

(D) None of the above

SHORT ANSWER QUESTIONS 02

(1) Derive the expression for length contraction using Lorentz transformation.

(2) A circular cylinder of radius 10 cm is obtained by sharply cutting a cube of edge 100 cm from its centre. No material is wasted. In a frame of reference where the cylinder and cube (after the cut) are at rest, the cylinder can thus slide in and out using a precision orientation mechanism. The cylinder is then subject to relative motion with a speed 99% of the speed of light. The theory of relativity demands that the moving body shall undergo a length contraction. In the frame of the cube the cylinder is in motion and hence cylinder will get contracted and therefore should effortlessly pass through the cube. On the other hand, in the frame of the cylinder, the cube is at relative motion and hence cube will undergo length contraction and hence the opening will become smaller than 10 cm and therefore the cylinder will not be able to pass through the opening.

These two are contradictory predictions. Draw a diagram showing the events in the problem and explain what exactly should happen as per the theory of relativity.

02-03: RELATIVISTIC ADDITION OF VELOCITIES

Fig 02-11: The total velocity of a kayak, like this one on the Deerfield River in Massachusetts, is its velocity relative to the water as well as the water's velocity relative to the riverbank. (credit: abkfenris, Flickr)

If you've ever seen a kayak move down a fast-moving river, you know that remaining in the same place would be hard. The river current pulls the kayak along. Pushing the oars back against the water can move the kayak forward in the water, but that only accounts for part of the velocity. The kayak's motion is an example of classical addition of velocities. In classical physics, velocities add as vectors. The kayak's velocity is the vector sum of its velocity relative to the water and the water's velocity relative to the riverbank.

Classical Velocity Addition

For simplicity, we restrict our consideration of velocity addition to one-dimensional motion. Classically, velocities add like regular numbers in one-dimensional motion. (See Figure 02-12) Suppose, for example, a girl is riding in a sled at a speed 1.0 m/s relative to an observer. She throws a snowball first forward, then backward at a speed of 1.5 m/s relative to the sled. We denote direction with plus and minus signs in one dimension; in this example, forward is positive. Let v be the velocity of the sled relative to the Earth, u the velocity of the snowball relative to the Earth-bound observer, and u′ the velocity of the snowball relative to the sled.

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Classical Velocity Addition

$$u=v+u'$$

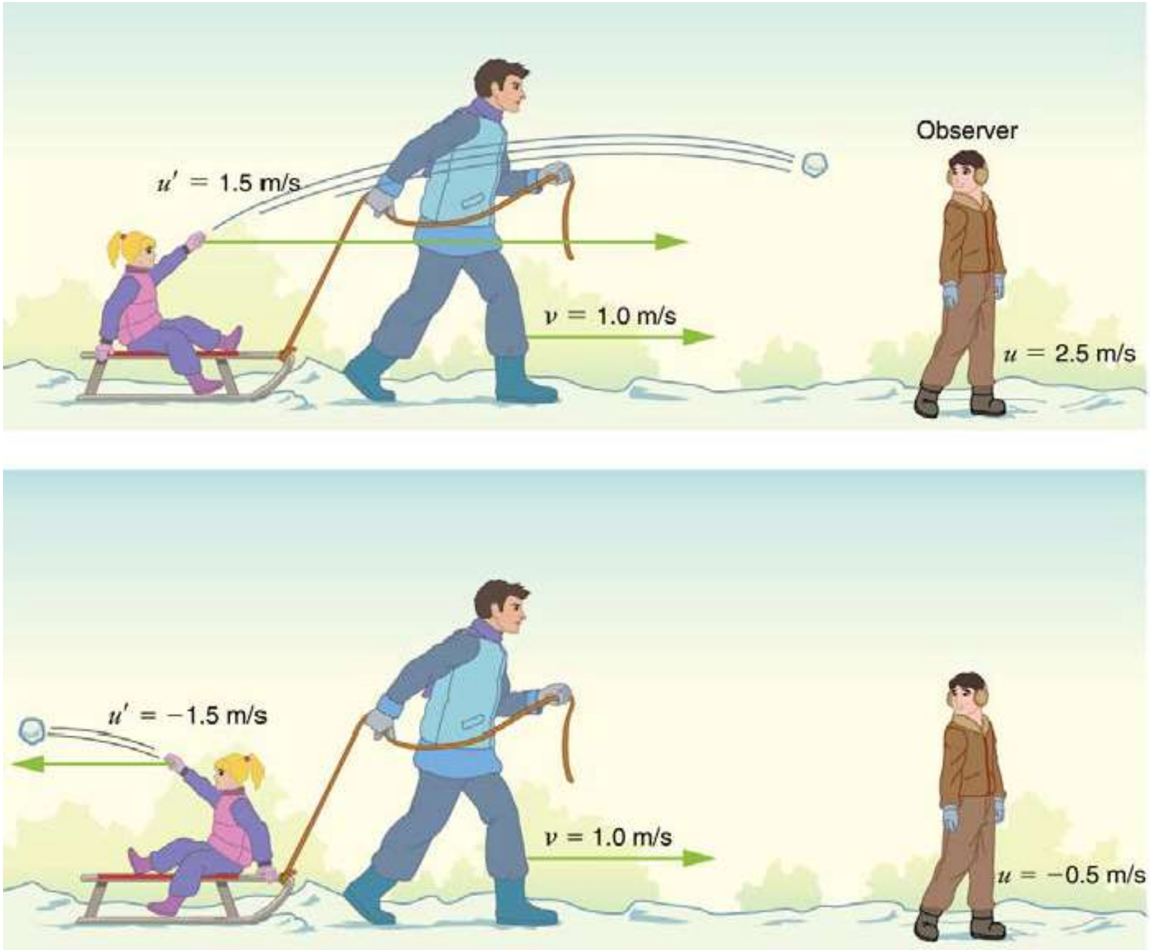

Fig 02-12: Classically, velocities add like ordinary numbers in one-dimensional motion. Here the girl throws a snowball forward and then backward from a sled. The velocity of the sled relative to the Earth is v=1.0 m/s . The velocity of the snowball relative to the truck is u′ , while its velocity relative to the Earth is u . Classically, u=v+u′.

Thus, when the girl throws the snowball forward, u = 1.0 m/s + 1.5 m/s = 2.5 m/s. It makes good intuitive sense that the snowball will head towards the earth-bound observer faster, because it is thrown forward from a moving vehicle. When the girl throws the snowball backward, u = 1.0 m/s +(− 1.5 m/s) = −0.5 m/s . The minus sign means the snowball moves away from the Earth-bound observer.

Relativistic Velocity Addition

The second postulate of relativity (verified by extensive experimental observation) says that classical velocity addition does not apply to light. Imagine a car traveling at night along a straight road, as in Figure. If classical velocity addition applied to light, then the light from the car's headlights would approach the observer on the sidewalk at a speed u= v+c . But we know that light will move away from the car at speed c relative to the driver of the car, and light will move towards the observer on the sidewalk at speed c, too.

Fig 02-13: According to experiment and the second postulate of relativity, light from the car's headlights moves away from the car at speed c and towards the observer on the sidewalk at speed c . Classical velocity addition is not valid.

Relativistic Velocity Addition

Either light is an exception, or the classical velocity addition formula only works at low velocities. The latter is the case. The correct formula for one-dimensional **relativistic velocity addition** is

$$u = \frac{v+u'}{1+\frac{vu'}{c^2}},$$

where v is the relative velocity between two observers, u is the velocity of an object relative to one observer, and u' is the velocity relative to the other observer. (For ease of visualization, we often choose to measure u in our reference frame, while someone moving at v relative to us measures u' .) Note that the term $\frac{vu'}{c^2}$ becomes very small at low velocities, and $u = \frac{v+u'}{1+\frac{vu'}{c^2}}$ gives a result very close to classical velocity addition. As before, we see that classical velocity addition is an excellent approximation to the correct relativistic formula for small velocities. No wonder that it seems correct in our experience.

Solved Problems

Problem 02-05 (Showing that speed of light in vacuum is constant irrespective of speed of frames of references)

Problem

Suppose a spaceship heading directly towards the earth at half the speed of light sends a signal to us on a laser-produced beam of light. Given that the light leaves the ship at speed c as observed from the ship, calculate the speed at which it approaches the Earth.

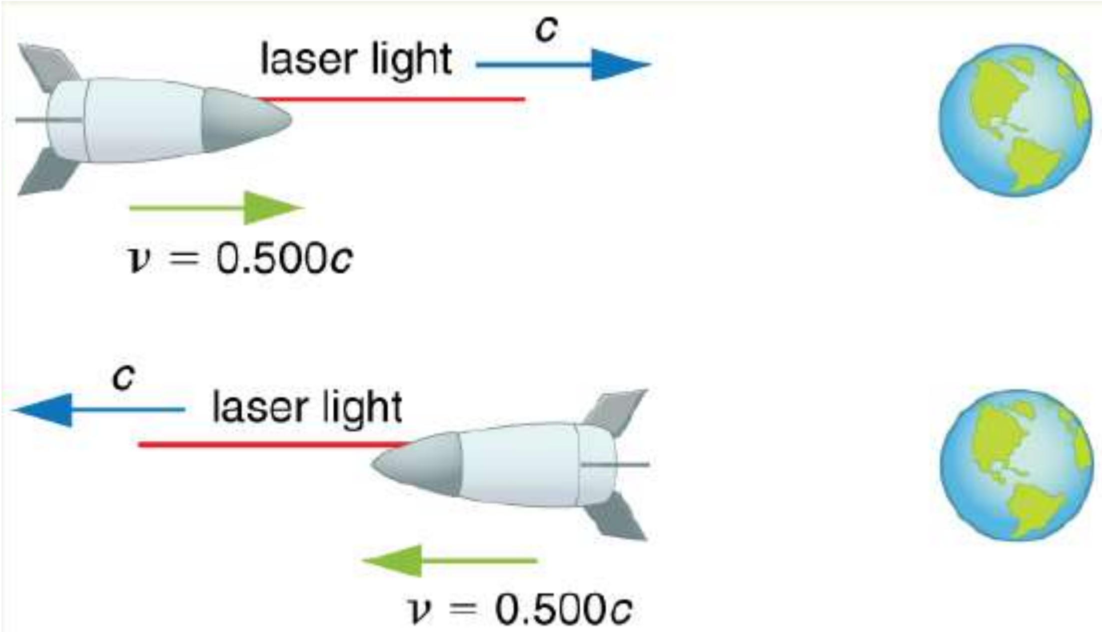

Fig 02-12: Problem figure

Strategy

Because the light and the spaceship are moving at relativistic speeds, we cannot use simple velocity addition. Instead, we can determine the speed at which the light approaches the Earth using relativistic velocity addition.

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Solution

1. Identify the knowns. $v = 0.500c$; $u' = c$
2. Identify the unknown. u
3. Choose the appropriate equation. $u = \dfrac{v + u'}{1 + \frac{vu'}{c^2}}$
4. Plug the knowns into the equation.

$$\begin{aligned} u &= \frac{v + u'}{1 + \frac{vu'}{c^2}} \\ &= \frac{0.500c + c}{1 + \frac{(0.500c)(c)}{c^2}} \\ &= \frac{(0.500 + 1)c}{1 + \frac{0.500c^2}{c^2}} \\ &= \frac{1.500c}{1 + 0.500} \\ &= \frac{1.500c}{1.500} \\ &= c \end{aligned}$$

Discussion

Relativistic velocity addition gives the correct result. Light leaves the ship at speed c and approaches the Earth at speed c. The speed of light is independent of the relative motion of source and observer, whether the observer is on the ship or earth-bound. Velocities cannot add to greater than the speed of light, provided that v is less than c and u′ does not exceed c. The following example illustrates that relativistic velocity addition is not as symmetric as classical velocity addition.

Problem 02-00 Comparing the Speed of Light towards and away from an Observer: Relativistic Package Delivery

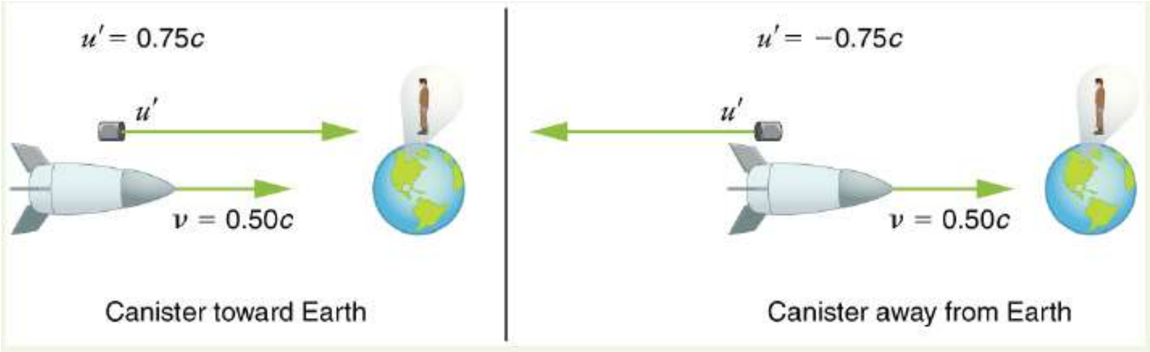

Fig 02-13: Figure for example

Suppose the spaceship in the previous example is approaching the earth at half the speed of light and shoots a canister at a speed of 0.750c .

(a) At what velocity will an Earth-bound observer see the canister if it is shot directly towards the Earth?

(b) If it is shot directly away from the Earth?

Strategy

Because the canister and the spaceship are moving at relativistic speeds, we must determine the speed of the canister by an Earth-bound observer using relativistic velocity addition instead of simple velocity addition.

Solution for (a)

1. Identify the knowns. $v=0.500c$; $u' = 0.750c$
2. Identify the unknown. u
3. Choose the appropriate equation. $u=\frac{v+u'}{1+\frac{vu'}{c^2}}$
4. Plug the knowns into the equation.

$$\begin{aligned} u &= \frac{v+u'}{1+\frac{vu'}{c^2}} \\ &= \frac{0.500c+0.750c}{1+\frac{(0.500c)(0.750c)}{c^2}} \\ &= \frac{1.250c}{1+0.375} \\ &= 0.909c \end{aligned}$$

Solution for (b)

1. Identify the knowns. $v = 0.500c$; $u' = -0.750c$
2. Identify the unknown. u
3. Choose the appropriate equation. $u = \frac{v+u'}{1+\frac{vu'}{c^2}}$
4. Plug the knowns into the equation.

$$\begin{aligned} u &= \frac{v+u'}{1+\frac{vu'}{c^2}} \\ &= \frac{0.500c + (-0.750c)}{1+\frac{(0.500c)(-0.750c)}{c^2}} \\ &= \frac{-0.250c}{1-0.375} \\ &= -0.400c \end{aligned}$$

Discussion

The minus sign indicates velocity away from the Earth (in the opposite direction from v), which means the canister is heading towards the Earth in part (a) and away in part (b), as expected. But relativistic velocities do not add as simply as they do classically. In part (a), the canister does approach the Earth faster, but not at the simple sum of 1.250c. The total velocity is less than you would get classically. And in part (b), the canister moves away from the Earth at a velocity of −0.400c, which is faster than the −0.250c you would expect classically. The velocities are not even symmetric. In part (a), the canister moves 0.409c faster than the ship relative to the Earth. In part (b) it moves 0.900c slower than the ship.

Doppler Shift

Although the speed of light does not change with relative velocity, the frequencies and wavelengths of light do. First discussed for sound waves, a Doppler shift occurs in any wave when there is relative motion between source and observer.

Pelativistic Doppler Effects

The observed wavelength of electromagnetic radiation is longer (called a red shift) than that emitted by the source when the source moves away from the observer and shorter (called a blue shift) when the source moves towards the observer.

$$=\lambda_{obs} = \lambda_s \sqrt{\frac{1+\frac{u}{c}}{1-\frac{u}{c}}}.$$

In the Doppler equation, λ_{obs} is the observed wavelength, λ_s is the source wavelength, and u is the relative velocity of the source to the observer. The velocity u is positive for motion away from an observer and negative for motion toward an observer. In terms of source frequency and observed frequency, this equation can be written

$$f_{\text{obs}} = f_s\sqrt{\frac{1-\frac{u}{c}}{1+\frac{u}{c}}}.$$

Problem 02

Suppose a galaxy is moving away from the Earth at a speed 0.825c. It emits radio waves with a wavelength of 0.525 m.

What wavelength would we detect on earth?

Strategy

Because the galaxy is moving at a relativistic speed, we must determine the Doppler shift of the radio waves using the relativistic Doppler shift instead of the classical Doppler shift.

Solution

1. Identify the knowns. $u=0.825c$; $\lambda_s = 0.525\ m$
2. Identify the unknown. λ_{obs}
3. Choose the appropriate equation. $\lambda_{\text{obs}} = \lambda_s\sqrt{\frac{1+\frac{u}{c}}{1-\frac{u}{c}}}$
4. Plug the knowns into the equation.

$$\begin{aligned}\lambda_{\text{obs}} &= \lambda_s\sqrt{\frac{1+\frac{u}{c}}{1-\frac{u}{c}}}\\ &= (0.525\ \text{m})\sqrt{\frac{1+\frac{0.825c}{c}}{1-\frac{0.825c}{c}}}\\ &= 1.70\ \text{m}.\end{aligned}$$

Discussion

Because the galaxy is moving away from the Earth, we expect the wavelengths of radiation it emits to be red-shifted. [Remember that the wavelength of red is about 7000 Angstrom or 700nm, while that of blue light is around 400nm. When the wavelength is changed towards higher wavelength, it is said to have red-shifted as red denotes the higher side of the spectrum. Similarly if the wavelength changes towards lower wavelength, we say it got blue-shifted] The wavelength we calculated is 1.70 m, which is redshifted from the original wavelength of 0.525 m.

The relativistic Doppler shift is easy to observe. This equation has everyday applications ranging from Doppler-shifted radar velocity measurements of transportation to Doppler-radar storm monitoring. In astronomical observations, the relativistic Doppler shift provides velocity information such as the motion and distance of stars.

Problem 02

Suppose a space probe moves away from the earth at a speed 0.350c. It sends a radio wave message back to the earth at a frequency of 1.50 GHz. At what frequency is the message received on the earth?

Solution

$$f_{\text{obs}} = f_S\sqrt{\frac{1-\frac{u}{c}}{1+\frac{u}{c}}} = (1.50\text{ GHz})\sqrt{\frac{1-\frac{0.350c}{c}}{1+\frac{0.350c}{c}}} = 1.04\text{ GHz}$$

Self-Test 28

(1) Suppose a galaxy is moving away from the earth at a speed 0.825 c. It emits radio waves with a wavelength of 0.525 m. What wavelength would we detect on the earth?

(A) 0.525 m

(B) 1.70 m

(C) 0.30 m

(D) 2.0 m

(2) Suppose the spaceship is approaching the earth at half the speed of light and shoots a canister at a speed of 0.750c. At what velocity will an earth-bound observer see the canister, if it is shot directly towards the earth?

(A) 1.25 c

(B) c

(C) 0.4 c

(D) 0.5c

Short Answer Questions 03

(1) What is the difference between velocity addition for non-relativist and relativist cases?

(2) Two muons are moving in the laboratory frames in opposite directions each with a speed equal to 55% the speed of light. What will be the speed of second muon with respect to the first muon?

Summary

- Relativity is the study of how different observers measure the same event.

• Modern relativity is divided into two parts. Special relativity deals with observers who are in uniform (unaccelerated) motion, whereas general relativity includes accelerated relative motion and gravity. Modern relativity is correct in all circumstances and, in the limit of low velocity and weak gravitation, gives the same predictions as classical relativity.

• An inertial frame of reference is a reference frame in which a body at rest remains at rest and a body in motion moves at a constant speed in a straight line unless acted on by an outside force.

• Modern relativity is based on Einstein's two postulates. The first postulate of special relativity is the idea that the laws of physics are the same and can be stated in their simplest form in all inertial frames of reference. The second postulate of special relativity is the idea that the speed of light c is a constant, independent of the relative motion of the source.

• The Michelson-Morley experiment demonstrated that the speed of light in a vacuum is independent of the motion of the Earth about the Sun.

Simultaneity and Time Dilation

• Two events are defined to be simultaneous if an observer measures them as occurring at the same time. They are not necessarily simultaneous to all observers—simultaneity is not absolute.

• Time dilation is the phenomenon of time passing slower for an observer who is moving relative to another observer.

• Observers moving at a relative velocity v do not measure the same elapsed time for an event. Proper time Δt_0 is the time measured by an observer at rest relative to the event being observed. Proper time is related to the time Δt measured by an Earth-bound observer by the equation

$$\Delta t = \frac{\Delta t_0}{\sqrt{1 - \frac{v^2}{c^2}}} = \gamma \Delta t_0,$$

Where

$$\gamma = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}.$$

• The equation relating proper time and time measured by an Earth-bound observer implies that relative velocity cannot exceed the speed of light.

• The twin paradox asks why a twin travelling at a relativistic speed away and then back towards the Earth ages less than the Earth-bound twin. The premise to the paradox is faulty because the traveling twin is accelerating. Special relativity does not apply to accelerating frames of reference.

This OpenStax book is available for free at http://cnx.org/content/col11406/1.9

• Time dilation is usually negligible at low relative velocities, but it does occur, and it has been verified by experiment.

Length Contraction

• All observers agree upon relative speed.

• Distance depends on an observer's motion. Proper length L_0 is the distance between two points measured by an observer who is at rest relative to both of the points. Earth-bound observers measure proper length when measuring the distance between two points that are stationary relative to the Earth.

• Length contraction L is the shortening of the measured length of an object moving relative to the observer's frame:

$$L=L_0\sqrt{1-\frac{v^2}{c^2}}=\frac{L_0}{\gamma}.$$

Relativistic Addition of Velocities

• With classical velocity addition, velocities add like regular numbers in one-dimensional motion: $u=v+u'$, where v is the velocity between two observers, u is the velocity of an object relative to one observer, and u' is the velocity relative to the other observer.

• Velocities cannot add to be greater than the speed of light. Relativistic velocity addition describes the velocities of an object moving at a relativistic speed:

$$u=\frac{v+u'}{1+\frac{vu'}{c^2}}$$

• An observer of electromagnetic radiation sees relativistic Doppler effects if the source of the radiation is moving relative to the observer. The wavelength of the radiation is longer (called a red shift) than that emitted by the source when the source moves away from the observer and shorter (called a blue shift) when the source moves toward the observer. The shifted wavelength is described by the equation

$$\lambda_{obs} = \lambda_s \sqrt{\frac{1 + \frac{u}{c}}{1 - \frac{u}{c}}}$$

λ_{obs} is the observed wavelength, λ_s is the source wavelength, and u is the relative velocity of the source to the observer.

References

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https://www.youtube.com/watch?v=pHfFSQ6pLGU (Yale University)

(Yale Univ., Lorenz transformation) https://www.youtube.com/watch?v=202fU9qIVK4

Velocity addition https://www.youtube.com/watch?v=F9BW57exRn8

Wikipedia

https://en.wikipedia.org/wiki/Special_relativity

https://en.wikipedia.org/wiki/Frame_of_reference

https://en.wikipedia.org/wiki/Lorentz_transformation

https://en.wikipedia.org/wiki/Twin_paradox

https://en.wikipedia.org/wiki/Relativistic_mechanics

https://en.wikipedia.org/wiki/Time_dilation

OER

Openstax, College Physics

Books

1. A P French, Special theory of relativeity

2. Feymann Lecture of Physics, 1962

3. Mechanics, Satya Prakash,

4. Mechanics, D.S. Mathus

5. REsnick and Halliday, Physics

Course Companion Website

Visit Here for Course Companion website for this course:

ANSWERS TO SELF-TESTS

Self Test 01	:	1	A	2	C										
Self Test 02	:	1	A	2	B										
Self Test 03	:	1	A	2	B										
Self Test 04	:	1	A	2	B										
Self Test 05	:	1	B	2	C										
Self Test 06	:	1	B	2	D										
Self Test 07	:	1	A	2	B										
Self Test 08	:	1	A	2	A										
Self Test 09	:	1	A	2	B										
Self Test 10	:	1	A	2	C										
Self Test 11	:	1	B	2	B	3	C	4	B	5	D	6	A	7	A
Self Test 12	:	1	C	2	B	3	C								
Self Test 13	:	1	A	2	B	3	D								
Self Test 14	:	1	C	2	B	3	D								
Self Test 15	:	1	A	2	B										
Self Test 16	:	1	A	2	B	3	A								
Self Test 17	:	1	A	2	A	3	B								
Self Test 18	:	1	C	2	D	3	B								
Self Test 19	:	1	A	2	D										
Self Test 20	:	1	A	2	C	3	A								
Self Test 21	:	1	B	2	A										
Self Test 22	:	1	C	2	A	3	B								
Self Test 23	:	1	D	2	A	3	D	4	B	5	C	6	D		
Self Test 24	:	1	C	2	D										
Self Test 25	:	1	C	2	A										
Self Test 26	:	1	D	2	A										
Self Test 27	:	1	B	2	C										
Self Test 28	:	1	B	2	C										

LABORATORY COMPONENT CREDIT 01

Measurements of length (or diameter) using Vernier caliper, screw gauge and travelling microscope.
To determine the Height of a Building using a Sextant.
To determine the Moment of Inertia of a Flywheel.
To determine the Young's Modulus of a Wire by Optical Lever Method.
To determine the Modulus of Rigidity of a Wire by Maxwell's needle.

LAB 01: MEASUREMENTS OF LENGTH (OR DIAMETER) USING VERNIER CALIPER, SCREW GAUGE AND TRAVELLING MICROSCOPE.

(A) VERNIER CALIPERS

Aim

Measurements of the various dimensions of an object using Vernier Calipers

OBJECTIVES

After completing the part of the Laboratory Activities you will be able to

- use Vernier Calipers for determining the various dimensions of an object.
- measure the diameter of a small spherical / cylindrical body.
- measure the length, width and height of the given rectangular block.
- measure the internal diameter and depth of a given beaker/calorimeter and hence find its volume.
- Explain the concept of least count etc for Vernier Calipers

Apparatus

Vernier Calipers, various objects to measure their dimentions like Internal Diameter, Outer Diameter, Depth, Length, etc

Description of Vernier calipers

Construction.

A caliper is a device used to measure the distance between two opposing sides of an object. It can be as simple as a compass with inward or outward-facing points. First the tips of the caliper are adjusted to fit across the points to be measured and the caliper is then removed and the distance between the tips is measured using a ruler.

The modern Vernier caliper was invented by Joseph R. Brown in 1851. It was the first practical tool for exact measurements that could be sold at an affordable price to ordinary machinists. The Vernier Caliper consists of a main scale fitted with a jaw at one end. Another jaw, containing the vernier scale, moves over the main scale. When the two jaws are in contact, the zero of the main scale and the zero of the Vernier scale should coincide. If both the zeros do not coincide, there will be a positive or negative zero error.

Parts of Vernier Calipers

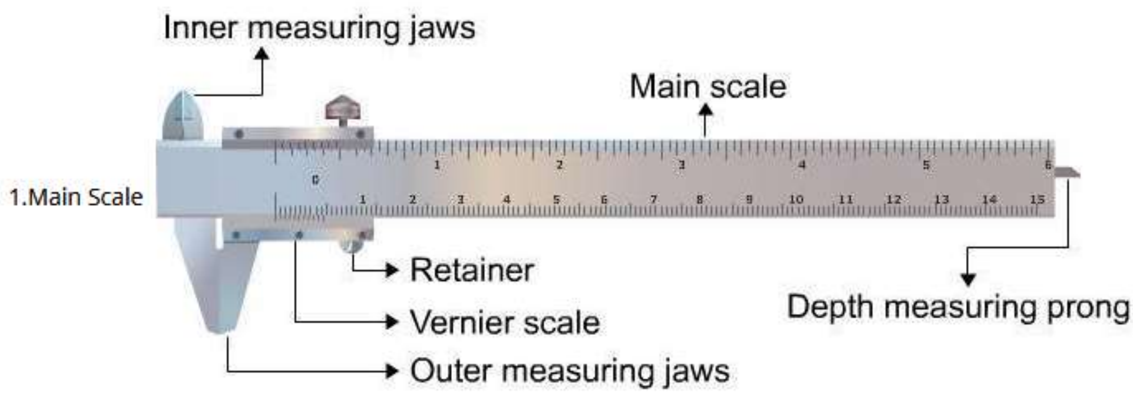

Fig 01.01 Parts of Vernier Calipers

1. The Main Scale

The main scale consists of a steel metallic strip graduated in centimeters at one edge and in inches at the other edge. It carries the inner and outer measuring jaws. When the two jaws are in contact, the zero of the main scale and the zero of the Vernier scale should coincide. If both the zeros do not coincide, there will be a positive or negative zero error.

2. Vernier Scale

A vernier scale slides on the strip. It can be fixed in any position by the retainer. On the Vernier scale, 0.9 cm is divided into ten equal parts.

3. Outer Measuring Calipers (Jaws)

The outer measuring jaws help to take the outer dimension of an object

4. Inner Measuring Jaws (Calipers)

The inner measuring jaws help to take the inner dimension of an object.

5. Retainer

The retainer helps to retain the object within the jaws of the Vernier calipers.

6. Depth Measuring Prong

The depth measuring prong helps to measure the depth of an object.

Least Count

The least count or the smallest reading which you can get with the instrument can be calculated as;

$$\text{Least count} = \text{one main scale (MS) division} - \text{one Vernier scale (VS) division.........(1)}$$

or

$$\text{Least Count} = \frac{\text{One Main scale(MS) division}}{\text{Number of divisions in Vernier Scale}}$$

First calculate the least count and only then place the object between the two jaws.

Record the position of zero of the Vernier scale on the main scale.

Calculating the Reading

When a body is between the jaws of the Vernier Caliper;

If the zero of the vernier scale lies ahead of the Nth division of the main scale, then the main scale reading (MSR) is;

$$MSR = N$$

If nth division of Vernier scale coincides with any division of the main scale, then the Vernier scale reading (VSR) is;

$$VSR = n \times L.C, \text{ (L.C is least count of vernier calliper)}$$

Total reading,

$$TR = MSR + VSR = N + (n \times L.C)\text{.........(2)}$$

Procedure

(A) Measuring the diameter of a small spherical or cylindrical body.

1. Keep the jaws of Vernier Callipers closed. Observe the zero mark of the main scale. It must perfectly coincide with that of the Vernier scale. If it doesn't coincide, account for the zero error for all observations to be made while using the instrument.

2. Look for the division on the vernier scale that coincides with a division of main scale. Use a magnifying glass, if available and note the number of division on the

Vernier scale that coincides with the one on the main scale. Position your eye directly over the division mark so as to avoid any parallax error.

3. Gently loosen the screw to release the movable jaw. Slide it enough to hold the sphere/cylindrical body gently (without any undue pressure) in between the lower jaws AB. The jaws should be perfectly perpendicular to the diameter of the body. Now, gently tighten the screw so as to clamp the instrument in this position to the body.

4. Carefully note the position of the zero mark of the vernier scale against the main scale. Usually, it will not perfectly coincide with any of the small divisions on the main scale. Record the main scale division just to the left of the zero mark of the vernier scale.

5. Start looking for exact coincidence of a vernier scale division with that of a main scale division in the vernier window from left end (zero) to the right. Note its number (say) N, carefully.

6. Multiply 'N' by least count of the instrument and add the product to the main scale reading noted in step 4. Ensure that the product is converted into proper units (usually cm) for addition to be valid.

7. Repeat steps 3-6 to obtain the diameter of the body at different positions on its curved surface. Take three sets of reading in each case.

8. Record the observations in the tabular form with proper units. Apply zero correction, if need be.

9. Find the arithmetic mean of the corrected readings of the diameter of the body. Express the results in suitable units with appropriate number of significant figures.

(B) MEASURING THE INTERNAL DIAMETER AND DEPTH OF THE GIVEN BEAKER (OR SIMILAR CYLINDRICAL OBJECT) TO FIND ITS INTERNAL VOLUME.

1. Adjust the upper jaws of the Vernier Callipers so as to touch the wall of the beaker from inside without exerting undue pressure on it. Tighten the screw gently to keep the Vernier Callipers in this position.

2. Repeat the steps 3-6 as in (a) to obtain the value of internal diameter of the beaker/calorimeter. Do this for two different (angular) positions of the beaker.

3. Keep the edge of the main scale of Vernier Callipers, to determine the depth of the beaker, on its peripheral edge. This should be done in such a way that the tip of the strip is able to go freely inside the beaker along its depth.

4. Keep sliding the moving jaw of the Vernier Callipers until the strip just touches the bottom of the beaker. Take care that it does so while being perfectly perpendicular to the bottom surface. Now tighten the screw of the Vernier Callipers.

5. Repeat steps 4 to 6 of part (a) of the experiment to obtain depth of the given beaker. Take the readings for depth at different positions of the breaker.

6. Record the observations in tabular form with proper units and significant figures. Apply zero corrections, if required.

7. Find out the mean of the corrected readings of the internal diameter and depth of the given beaker. Express the result in suitable units and proper significant figures.

OBSERVATIONS

(I)Least count of Vernier Callipers (Vernier Constant)

1 main scale division (MSD) = …. mm = …. cm

Number of vernier scale divisions, N = ….

10 vernier scale divisions = …. main scale divisions

1 vernier scale division = ….. main scale division

Vernier constant

= 1 main scale division – 1 vernier scale division

= …………….main scale divisions

= ……………. main scale division

Vernier constant (V_C)

= …………….mm = …………….cm

Alternatively, 1MSD

Vernier constant= $\frac{1\,MSD}{N}$ =…………….

Vernier constant (V_C) = ……………. mm = ……………. cm

(II) Zero error and its correction

When the jaws A and B touch each other, the zero of the Vernier should coincide with the zero of the main scale. If it is not so, the instrument is said to possess zero error (e). Zero error may be positive or negative, depending upon whether the zero of vernier scale lies to the right or to the left of the zero of the main scale. This is shown by the Fig. In this situation, a correction is required to the observed readings

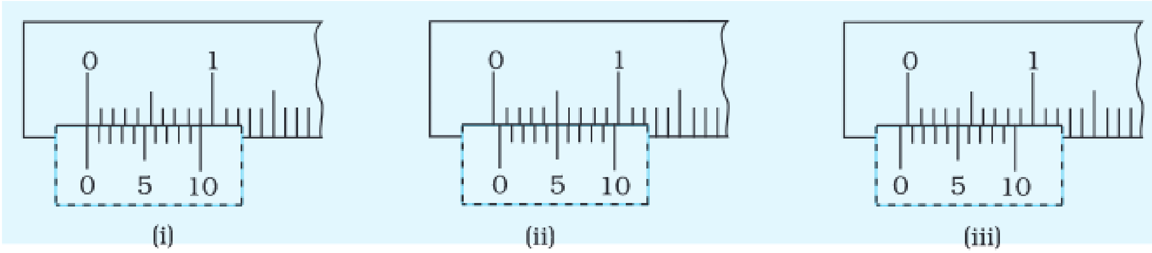

Fig 01-02: (i) No Zero error (ii)Positive zero error (iii)Negative zero error

(III) Positive zero error

Fig (ii) shows an example of positive zero error. From the figure, one can see that when both jaws are touching each other, zero of the vernier scale is shifted to the right of zero of the main scale (This might have happened due to manufacturing defect or due to rough handling). This situation makes it obvious that while taking measurements, the reading taken will be more than the actual reading. Hence, a correction needs to be applied which is proportional to the right shift of zero of vernier scale.

In ideal case, zero of vernier scale should coincide with zero of main scale. But in Fig. (ii), 5th vernier division is coinciding with a main scale reading.

$\therefore$ Zero Error = + 5 × Least Count = + 0.05 cm

Hence, the zero error is positive in this case. For any measurements done, the zero error (+ 0.05 cm in this example) should be 'subtracted' from the observed reading.

$\therefore$ True Reading = Observed reading – (+ Zero error)

(IV) Negative zero error

Fig. 01-00 (iii) shows an example of negative zero error. From this figure, one can see that when both the jaws are touching each other, zero of the vernier scale is shifted to the left of zero of the main scale. This situation makes it obvious that while taking measurements, the reading taken will be less than the actual reading. Hence, a correction needs to be applied which is proportional to the left shift of zero of vernier scale.

In Fig. 01-00 (iii), 5^{th} vernier scale division is coinciding with a main scale reading.

$\therefore$ Zero Error = – 5 × Least Count = – 0.05 cm

Note that the zero error in this case is considered to be negative.

For any measurements done, the negative zero error, (–0.05 cm in this example) is also substracted 'from the observed reading', though it gets added to the observed value.

$\therefore$ True Reading = Observed Reading – (– Zero error)

Sr No	Main Scale Reading M (cm)	No of coinciding division on Vernier N	Vernier scale reading, $V = N \times V_c$ (cm)	Measured Diameter M+V (cm)

Zero error, e = ± … cm

Mean observed diameter = … cm

Corrected diameter = Mean observed diameter – Zero Error

Result

The dimension of the given object is

……………….. cm

Precautions

1. If the vernier scale is not sliding smoothly over the main scale, apply machine oil/grease.

2. Screw the vernier tightly without exerting undue pressure to avoid any damage to the threads of the screw.

3. Keep the eye directly over the division mark to avoid any error due to parallax.

4. Note down each observation with correct significant figures and units

Sources of error

Any measurement made using Vernier Calipers is likely to be incorrect if -

(i) the zero error in the instrument placed is not accounted for; and

(ii) the Vernier Calipers is not in a proper position with respect to the body, avoiding gaps or undue pressure or both

(B) SCREW GAUGE (MICROMETER)

AIM

To use screw gauge to measure various dimensions (like thickness or radius) of an object

OBJECTIVES

Use of screw gauge to

(a) measure diameter of a given wire,

(b) measure thickness of a given sheet; and

(c) determine volume of an irregular lamina

APPARATUS AND MATERIAL

Screw gauge, wires, laminar sheets, and other material

DESCRIPTION OF SCREW GAUGE

With Vernier Callipers. you are usually able to measure length accurately up to 0.1 mm. More accurate measurement of length, upto 0.01 mm or 0.005 mm, may be made by using a screw gauge. As such a Screw Gauge is an instrument of higher precision than a Vernier Callipers. You might have observed an ordinary screw as in Figure 01-03. There are threads on a screw. The separation between any two consecutive threads is the same. The screw can be moved backward or forward in its nut by rotating it anti-clockwise or clockwise.

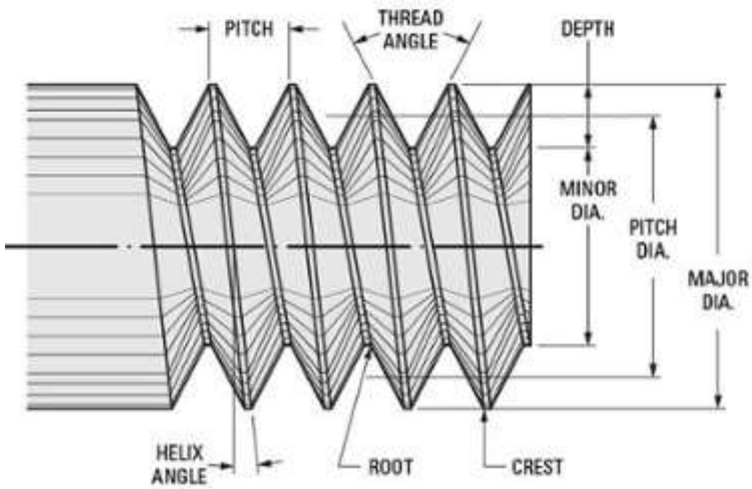

Fig 01-03: Pitch of a screw is the distance between successive threads

The distance advanced by the screw when it makes its one complete rotation is the separation between two consecutive threads. This distance is called the Pitch of the screw. Fig 01-04 shows the pitch of the screw. It is usually 1 mm or 0.5 mm.

Fig 01.04 Micrometer (screw gauge) showing reading 1.639 mm ± 0.005.

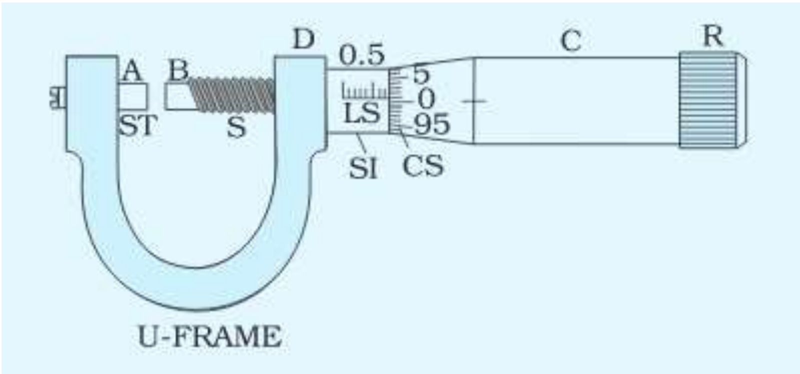

Fig 01-05: Various parts of a micrometer (simplified diagram)

Fig. 01-05 shows a a simplified diagram of the screw gauge. It has a screw 'S' which advances forward or backward as one rotates the head C through rachet R. There is a linear scale 'LS' attached to limb D of the U frame. The smallest division on the linear scale is 1 mm (in one type of screw gauge). There is a circular scale CS on the head, which can be rotated. There are 100 divisions on the circular scale. When the end B of the screw touches the surface A of the stud ST, the zero marks on the main scale and the circular scale should coincide with each other.

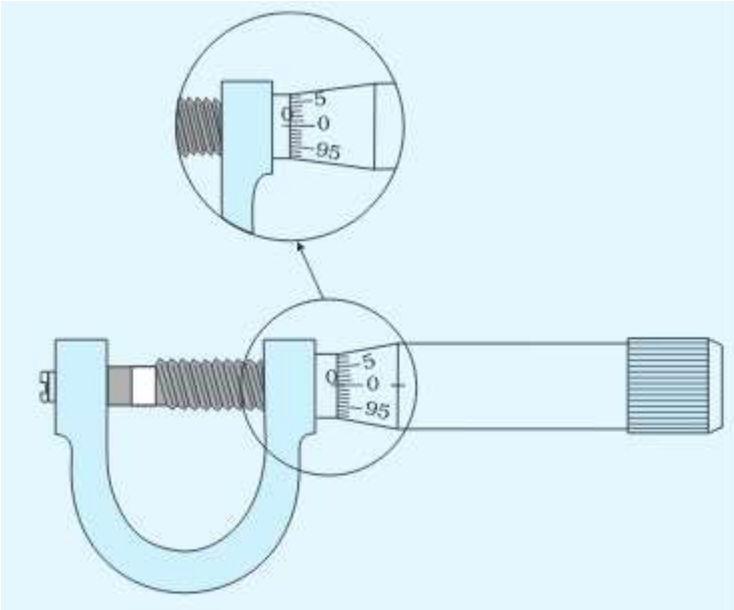

Fig 01.06: A screw gauge with no zero error.

ZERO ERROR

When the end of the screw and the surface of the stud are in contact with each other, the linear scale and the circular scale reading should be zero. In case this is not so, the screw gauge is said to have an error called zero error.

Fig. 01-06 shows an enlarged view of a screw gauge with its faces A and B in contact. Here, the zero mark of the LS and the CS are coinciding with each other.

When the reading on the circular scale across the linear scale is more than zero (or positive), the instrument has Positive zero error as shown in Fig. 01-00 (a). When the reading of the circular scale across the linear scale is less than zero (or negative), the instrument is said to have negative zero error as shown in Fig. 01-00 (b).

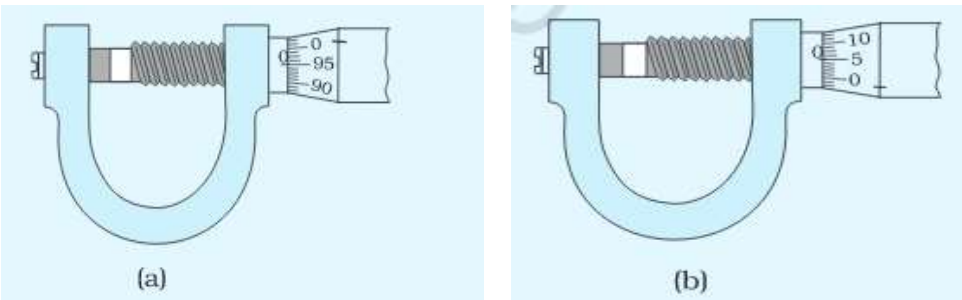

Fig 01-07: Screw gauge shows positive zero error in (a) and negative zero error in (b)

TAKING THE LINEAR SCALE READING

The mark on the linear scale which lies close to the left edge of the circular scale is the linear scale reading. For example, the linear scale reading as shown in Fig., is 0.5 cm.

TAKING CIRCULAR SCALE READING

The division of circular scale which coincides with the main scale line is the reading of circular scale. For example, in the Fig., the circular scale reading is 2.

TOTAL READING

Total reading

= linear scale reading + circular scale reading × least count

= 0.5 + 2 × 0.001

= 0.502 cm

THEORETICAL PRINCIPLES

The linear distance moved by the screw is directly proportional to the rotation given to it. The linear distance moved by the screw when it is rotated by one division of the circular scale, is the least distance that can be measured accurately by the instrument. It is called the least count of the instrument.

$$Least\ Count = \frac{pitch}{No\ of\ division\ on\ the\ circular\ scale}$$

For example for a screw gauge with a pitch of 1mm and 100 divisions on the circular scale,the least count is 1 mm/100 = 0.01 mm $=10^{-5}$ m.

This is the smallest length one can measure with this screw gauge.

Fig 01-08: Micrometer with a vernier scale. Vernier micrometer reading 5.783 mm ± 0.001 mm. made up from 5.5mm on main screw lead scale, 0.28mm on screw rotation scale, and 0.003mm added from vernier.

In another type of screw gauge, pitch is 0.5 mm and there are 50 divisions on the circular scale. The least count of this screw gauge is 0.5 mm/50 = 0.01 mm. Note that here two rotations of the circular scale make the screw to advance through a distance of 1 mm. Some screw gauge have a least count of 0.001 mm or one micrometer (i.e. 10^{-6} m) and therefore are called micrometer screw. Such screw gauges may have a vernier scale as shown in the figure 1.09 given below.

Some micrometers (called indicating micrometers) have a pressure sensor attached to them so that you apply precisely a predetermined pressure when holding the object within the 'jaws'. This eliminates subjectivity. Otherwise one observer will squeeze the object too much and find the thickness of the object as too little while another may barely hold the object and get the thickness overestimated.

Fig 01-09: Indicating micrometer eliminates the subjectivity.

PROCEDURE OF MEASUREMENT OF DIAMETER OF A GIVEN WIRE

1. Take the screw gauge and make sure that the rachet R on the head of the screw functions properly.

2. Rotate the screw through, say, ten complete rotations and observe the distance through which it has receded. This distance is the reading on the linear scale marked by the edge of the circular scale. Then, find the pitch of the screw, i.e., the distance moved by the screw in one complete rotation. If there are n divisions on the circular scale, then distance moved by the screw when it is rotated through one division on the circular scale is called the least count of the screw gauge, that is,

$$Least\ Count = \frac{pitch}{No\ of\ division\ on\ the\ circular\ scale}$$

3. Insert the given wire between the screw and the stud of the screw gauge. Move the screw forward by rotating the rachet till the wire is gently gripped between the screw and the stud. Stop rotating the rachet the moment you hear a click sound.

4. Take the readings on the linear scale and the circular scale.

5. From these two readings, obtain the diameter of the wire.

6. The wire may not have an exactly circular cross-section. Therefore, it is necessary to measure the diameter of the wire for two positions at right angles to each other. For this, first record the reading of diameter d1 and then rotate the wire through 90° at the same cross-sectional position. Record the reading for diameter d2 in this position.

7. The wire may not be truly cylindrical. Therefore, it is necessary to measure the diameter at several different places and obtain the average value of diameter. For this, repeat the steps (3) to (6) for three more positions of the wire.

8. Take the mean of the different values of diameter so obtained.

9. Subtract zero error, if any, with proper sign to get the corrected value for the diameter of the wire.

Observations and Calculations

The length of the smallest division on the linear scale = ………. Mm

Distance moved by the screw when it is rotated through x complete rotations, y = ………. Mm

Pitch of the screw = $\frac{y}{x} = $ = ………. Mm

Number of divisions on the circular scale n = ……….

Least Count (L.C.) of screw guage

$$Least\ Count = \frac{pitch}{No\ of\ division\ on\ the\ circular\ scale}$$

No. of divisions on the circular scale

= ……….mm

Zero error with sign (No. of div. × L. C.) = ……….mm

Sr No	Reading in one direction (d1)			Reading in perpendicular direction (d2)			Measured
	Linear Scale Reading M (cm)	Circular Scale Reading (n)	Diameter d1 = M + n × Least Count (cm)	Linear Scale Reading M (cm)	Circular Scale Reading (n)	Diameter d2 = M + n × Least Count (cm)	Dia (d1+d2)/2 (cm)

Mean diameter = ... mm

Mean corrected value of diameter

= measured diameter – (zero error with sign) = ... mm

RESULT

The diameter of the given wire as measured by screw gauge is ... m.

PRECAUTIONS

1. Rachet arrangement in screw gauge must be utilized to avoid undue pressure on the wire as this may change the diameter.

2. Move the screw in one direction else the screw may develop “play”.

3. Screw should move freely without friction.

4. Reading should be taken at least at four different points along the length of the wire.

5. View all the reading keeping the eye perpendicular to the scale to avoid error due to parallax.

SOURCES OF ERROR

1. The wire may not be of uniform cross-section.

2. Error due to backlash though can be minimized but cannot be completely eliminated.

REFERENCE

http://amrita.olabs.edu.in/?sub=1&brch=5&sim=16&cnt=1

http://ncert.nic.in/ncerts/l/kelm102.pdf

(C) USING TRAVELLING MICROSCOPE TO MEASURE DIAMETER OF A CAPILLARY TUBE

AIM:

To find diameter of a capillary tube using travelling microscope

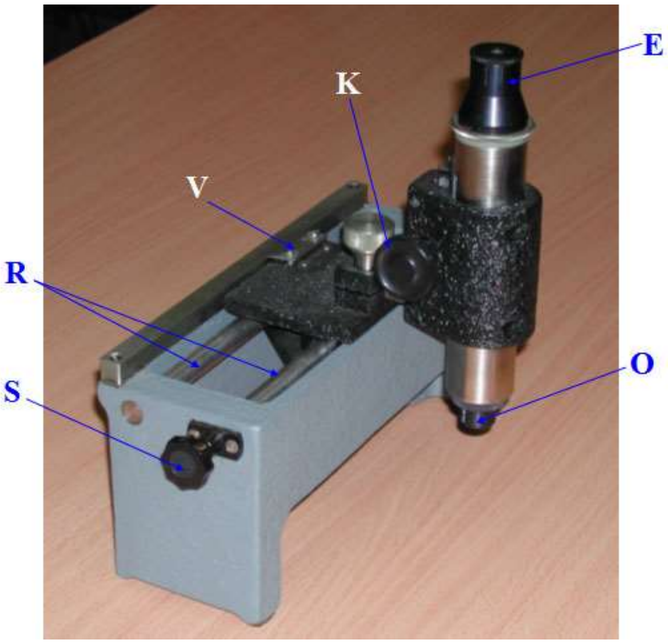

Fig 01-10: A traveling microscope and its parts: E-eyepiece, O-objective, K-knob for focusing, V-vernier, R—rails, S—screw for fine position adjustment.

OBJECTIVES

After completing this Lab part, you will be able to

- use traveling microscope to find dimensions of fine objects without disturbing them to a precision of around 0.01 mm.

THEORY

A travelling microscope is an instrument for measuring length with a resolution typically in the order of 0.01mm. The precision is such that better-quality instruments have measuring scales made from Invar to avoid misreading due to thermal effects. The instrument comprises a microscope mounted on two rails fixed to, or part of a very rigid bed. The position of the microscope can be varied coarsely by sliding along the rails, or finely by turning a screw. The eyepiece is fitted with fine cross-hairs to fix a precise position, which is then read off the vernier scale. Some instruments, such as that produced in the 1960s by the Precision Tool and Instrument Company of Thornton

Heath, Surrey, England, also measure vertically. The purpose of the microscope is to aim at reference marks with much higher accuracy than is possible using the naked eye. It is used in laboratories to measure the refractive index of liquids using the geometrical concepts of ray optics. It is also used to measure very short distances precisely, for example the diameter of a capillary tube. This mechanical instrument has now largely been superseded by electronic- and optically-based measuring devices that are both very much more accurate and considerably cheaper to produce.

Travelling microscope consists of a cast iron base with machined-Vee-top surface and is fitted with three leveling screws. A metallic carriage, clamped to a spring-loaded bar slides with its attached vernier and reading lens along an inlaid strip of metal scale. The scale is divided in half millimeters. Fine adjustments are made by means of a micrometer screw for taking accurate reading. Microscope tube consists of 10 x Eyepice and 15mm or 50mm or 75mm objectives. The Microscope, with its rack and pinion attachment is mounted on a vertical slide, which too, runs with an attached vernier along the vertical scale. The microscope is free to rotate in vertical plane. The vertical guide bar is coupled to the horizontal carriage of the microscope. For holding objects a horizontal stage made of a milki conolite sheet is provided in the base.

PROCEDURE

DETERMINATION OF DIAMETER

FOR DETERMINATION OF DIAMETER OF THE CAPILLARY ALONG THE HORIZONTAL DIRECTION

1. Mount the capillary tube in horizontal direction in a stand with the help of a rubber cork to place and hold the capillary tube. Rotate the microscope so that it is horizontal and in line with the tip of the capillary tube.

2. Now looking through the microscope, turn the focussing screw to get a clear image of the capillary tube. Now adjust the microscope in such a way that the vertical crosswire coincides with the left end of the capillary tube.

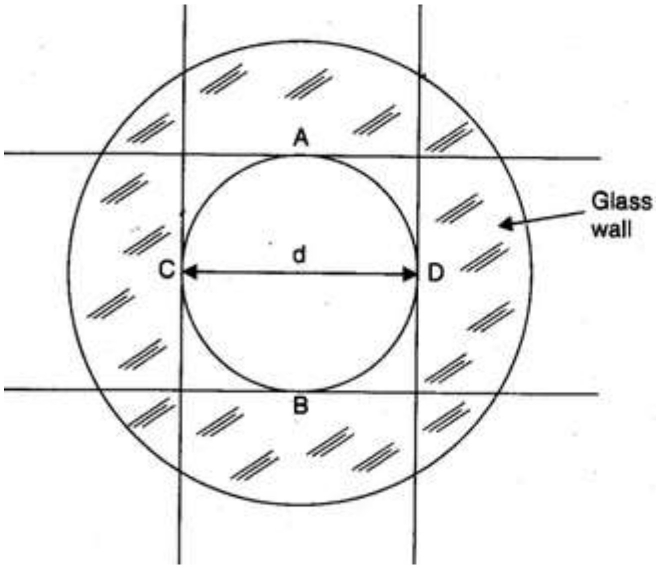

Fig 01-11: View from traveling microscope

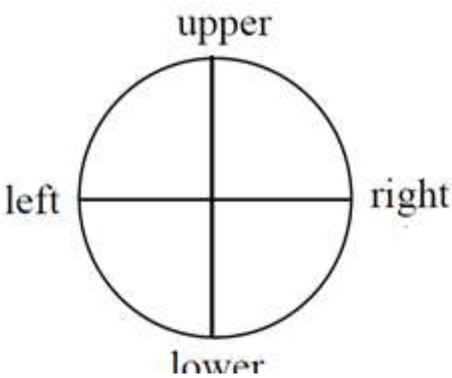

Fig 01-12: crosswire as seen from eyepiece.

3. In the horizontal scale, look the zero of the vernier, and find out the division on the main scale just before the zero mark. Note it as the MSR.

4. Now look carefully at the vernier. Any one of the fifty lines will come exactly in line with one of the lines of the main scale. That division on the vernier is the vernier scale reading. Note it down in the observation table.

5. Now move the telescope horizontally to focus on the right end of the capillary tube. Again take the reading as before.

6. Repeat the experiment by moving the telescope vertically, coinciding the horizontal crosswire with top and bottom and now the readings are taken on the vertical scale.

7. From the observations you will get two values of diameter, one for vertical and one for the horizontal.

OBSERVATIONS

Least count of the travelling microscope = …… cm

Sr No	Crosswire at the left end of capillary (I)			Crosswire at the right end of capillary (II)			Measured
	Main Scale Reading M S R (cm)	Vernier Scale Div VSD (n)	Total Reading TR (I) = MSR + VSD × L.C. (cm)	Main Scale Reading M S R (cm)	Vernier Scale Div VSD (n)	Total Reading TR (II) = MSR + VSD × L.C. (cm)	Dia II - I (cm)
						MEAN DIA =	

RESULT

The diameter of the capillary tube = …….. cm (within experimental error …….. cm)

PRECAUTIONS

1. Be careful to not to disturb the capillary between two readings.

2. Be careful not to disturb the focus or vertical positions between readings.

3. Be careful to avoid parallax

SOURCES OF ERROR

1. Microscope may not be properly calibrated.
2. The vertical distance between left and right side of capillary and microscope may be different if microscope is not mounted truly vertically using spirit level.

REFERENCES

1. Wikpedia, **https://en.wikipedia.org/wiki/Traveling_microscope**

2. http://www.plustwophysics.com/physics-practical-measurement-of-diameter-of-capillary-tube-using-travelling-microscope/

LAB 02: TO DETERMINE THE HEIGHT OF A BUILDING USING A SEXTANT.

Aim

To determine height of a building using a sextant.

Apparatus

A sextant, 50 m measuring tape, plumb line, spirit level, sextant stand with rigid clamp, a piece of chalk or marked arrow heads on board

THEORY

The height of a building may be easier to find if the distance from the foot to any point of observation can be determined. In such case the building is said to be accessible.

When the foot of the object like hill or a building cannot be reached directly and distance between the foot and the sextant cannot be measured, it is said to be inaccessible. Let us see how we can measure the height of an accessible and inaccessible object.

As seen from the figure, OA is the height of the building also denoted by h. We may take measurements of angle subtended by the building at three points B,C and D. In fact at least two points are needed if the building is inaccessible and at least one point of observation is needed if the building is accessible. More points of observation helps us determine the height more precisely and reliably as we may take average and minimize error.

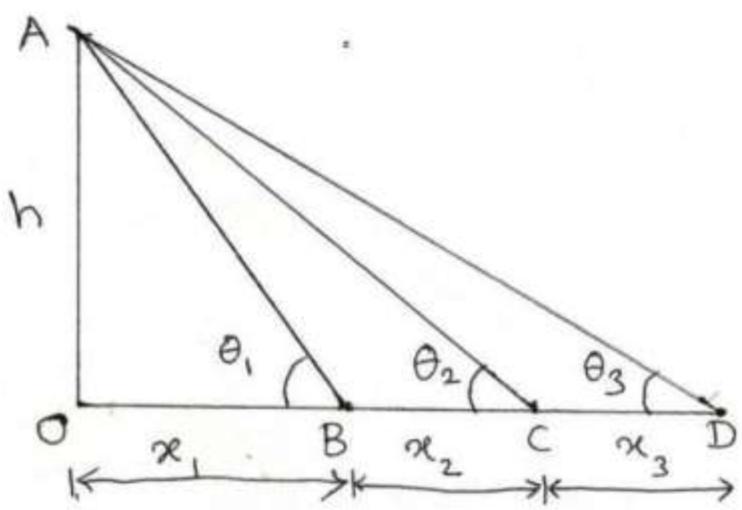

Fig 02-01: Finding height of a structure using measurements of angles and distances

From the figure, we note that

$$h = x_1 \tan\theta_1$$

$$h = (x_1 + x_2)\tan\theta_2$$

$$h = (x_1 + x_2 + x_3)\tan\theta_3$$

If the building is accessible, we can find the values of $x_1,\ x_2\ and\ x_3$ as well as the three anglesθ_1, θ_2 and θ_3. Thus we get three values of h and we can find the mean to get more reliable value.

If the object is inaccessible we may not be able to find x_1 as we may not be able to reach the 'foot' of the object. However, $x_2\ and\ x_3$ will be known.

We find height by eliminating x1 from the following equations

$$x_1 = h\cot\theta_1$$

$$x_1 + x_2 = h\ \cot\theta_2$$

$$x_1 + x_2 + x_3 = h\ \cot\theta_3$$

Hence,

$$x_2 = h\ \cot\theta_2 - h\ \cot\theta_1$$

$$h = \frac{x_2}{\cot\theta_2 - \ \cot\theta_1}$$

Similarly

$$x_2 = h\ \cot\theta_3 - h\ \cot\theta_2$$

$$h = \frac{x_3}{\cot\theta_3 - \ \cot\theta_2}$$

From these we can find two values of h and we can get a more reliable value for height by obtaining the mean of these two values.

$$d = x + y = hcot\,\theta_2$$

$$y = hcot\,\theta_1$$

$$x = d - y = hco\,t\,\theta_2 - hco\,t\,\theta_1 = h(co\,t\,\theta_2 - co\,t\,\theta_1)$$

$$h = \frac{x}{h(co\,t\,\theta_2 - co\,t\,\theta_1)}$$

Now x can be measured with the help of measuring tape, therefore h can be calculated.

ABOUT SEXTANT

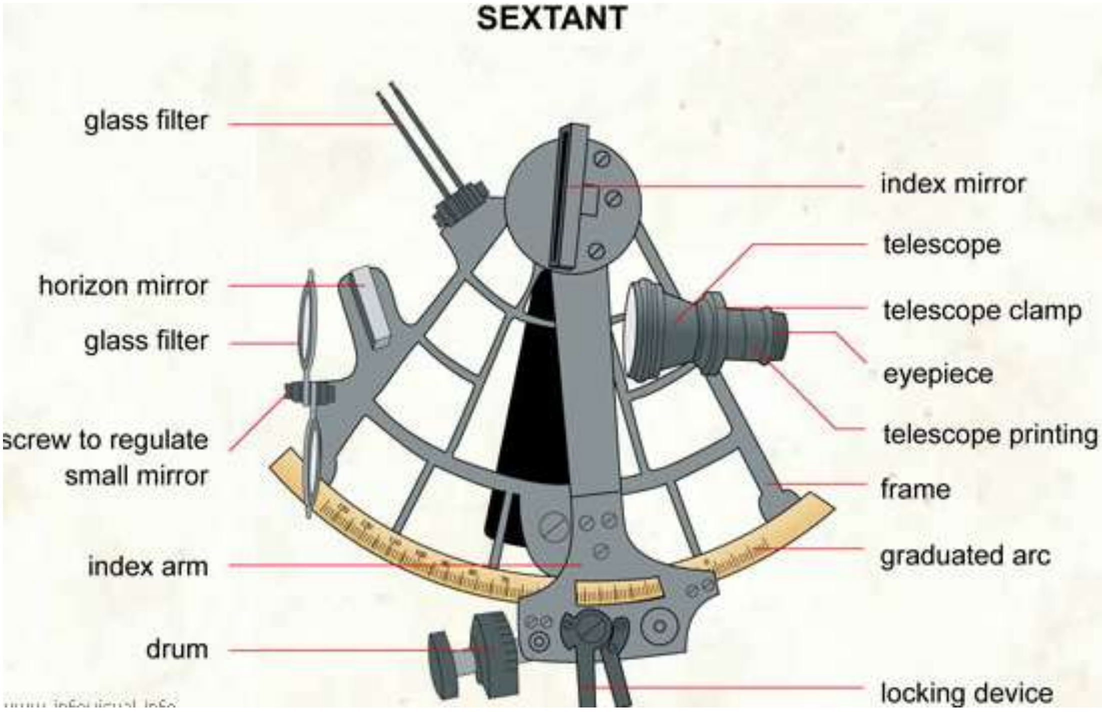

Fig 02-02: Parts of a sextant

A sextant is a doubly reflecting navigation instrument that measures the angular distance between two visible objects. The primary use of a sextant is to measure the angle between an astronomical object and the horizon for the purposes of celestial navigation. The estimation of this angle, the altitude, is known as sighting or shooting the object, or taking a sight. The angle, and the time when it was measured, can be used to calculate a position line on a nautical or aeronautical chart—for example, sighting the Sun at noon or Polaris at night (in the Northern Hemisphere) to estimate latitude. Sighting the height of a landmark can give a measure of distance off and, held horizontally, a sextant can measure angles between objects for a position on a chart. A sextant can also be used to measure the lunar distance between the moon and another celestial object (such as a star or planet) in order to determine Greenwich Mean Time and hence longitude. The principle of the instrument was first implemented around 1731 by John Hadley (1682–1744) and Thomas Godfrey (1704–1749), but it was also found later in the unpublished writings of Isaac Newton (1643–1727). Additional links can be found to Bartholomew Gosnold (1571–1607) indicating that the use of a sextant for nautical navigation predates Hadley's implementation. In 1922, it was modified for aeronautical navigation by Portuguese navigator and naval officer Gago Coutinho.

ADJUSTMENT

Due to the sensitivity of the instrument it is easy to knock the mirrors out of adjustment. For this reason a sextant should be checked frequently for errors and adjusted accordingly.

There are four errors that can be adjusted by the navigator and they should be removed in the following order.

Perpendicularity error

This is when the index mirror is not perpendicular to the frame of the sextant. To test for this, place the index arm at about 60° on the arc and hold the sextant horizontally with the arc away from you at arms length and look into the index mirror. The arc of the sextant should appear to continue unbroken into the mirror. If there is an error, then the two views will appear to be broken. Adjust the mirror until the reflection and direct view of the arc appear to be continuous.

Side error

This occurs when the horizon glass/mirror is not perpendicular to the plane of the instrument. To test for this, first zero the index arm then observe a star through the sextant. Then rotate the tangent screw back and forth so that the reflected image passes alternately above and below the direct view. If in changing from one position to another, the reflected image passes directly over the unreflected image, no side error exists. If it passes to one side, side error exists. The user can hold the sextant on its side and observe the horizon to check the sextant during the day. If there are two horizons there is side error; adjust the horizon glass/mirror until the stars merge into one image or the horizons are merged into one. Side error is generally inconsequential for observations and can be ignored or reduced to a level that is merely inconvenient.

Collimation error

This is when the telescope or monocular is not parallel to the plane of the sextant. To check for this you need to observe two stars 90° or more apart. Bring the two stars into coincidence either to the left or the right of the field of view. Move the sextant slightly so that the stars move to the other side of the field of view. If they separate there is collimation error. As modern sextants rarely use adjustable telescopes, they do not need to be corrected for collimation error.

Index error

This occurs when the index and horizon mirrors are not parallel to each other when the index arm is set to zero. To test for index error, zero the index arm and observe the horizon. If the reflected and direct image of the horizon are in line there is no index error. If one is above the other adjust the index mirror until the two horizons merge. This can be done at night with a star or with the moon.

Procedure

1. Mark an arrow head as reference point at the distant object for which the height is to be determined. Adjust the height of the axis of the telescope of sextant is clamped in a vertical position the direct point of this reference mark is seen

through the telescope. The height of the top of the object is to be determined with respect to this mark. Make the scale vertical with plump line.

2. Move the index arm till the direct and reflected images of the reference arrow overlap each other and are equally bright. For the correctly adjusted sextant this will happen when the zero of vernier coincides with nearly the zeroes of the main scale.

3. Now move the index arm so that half of the field of views moves past the stationary images of the fixed reference mark. Go on moving till the reflected image of upper point i.e. top of the object coincides with the direct image of the arrow.

4. Move the sextant either forward or backward through a known distance ‘x’ i.e. the centers of these two points are in the same straight line.

5. Repeat the experiment for a third point D.

Observation

NO.	Position of sextant	Sextant scale	reading	Diff. b-a= θ	cot θ	h
		Initial reading=a	Final reading=b			
1. (B)						
2. (C)						

3. (D)						

CALCULATIONS

The building is accessible / inaccessible

Find the two/three values of height using your observations

Find the height of the building by taking the mean.

RESULT

The height of the building is =…………………………… m.

REFERENCE

1. Wikipedia, **https://en.wikipedia.org/wiki/Sextant**

LAB 03: TO DETERMINE THE MOMENT OF INERTIA OF A FLYWHEEL.

Aim

To study the angular motion of flywheel and to determine its moment of inertia.

Apparatus

a flywheel, string, stopwatch, weights

THEORY

A flywheel is a disc with a horizontal axle passing through its center of mass. The two ends of the axle are mounted with bearings as shown in Fig. 03-01. This arrangement is kept at a distance H from the ground level. A string of length just equal to the height H is taken. The string is wound tightly and uniformly on the axle of the flywheel by hooking its one end to a vertical pin fixed to the axle of the flywheel. A mass m is attached to the other end of the string. If the mass is allowed to fall, it unwinds the string and sets the flywheel into rotational motion. When the string unwinds completely, it detaches from the axle and the mass falls to the ground. The flywheel continues to rotate a few more turns due to its rotational inertia and comes to rest after some time due to its frictional force.

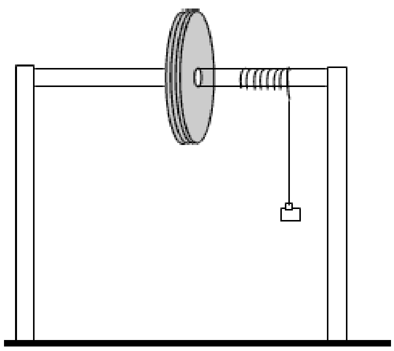

Fig 03-01: Experimental setup.

Theory:

In this experiment, the potential energy of mass m is converted into its translation kinetic energy and rotational kinetic energy of flywheel and some of the energy is lost in overcoming frictional force. The conservation of energy equation at the instant when the mass touches the ground can be written as,

$$mgH = \frac{1}{2}mv^2 + \frac{1}{2}I\omega^2 + n_1 f \qquad (03\text{-}01)$$

Here v is the velocity of mass and ω is the angular velocity of flywheel at the instant when the mass touches the ground. Here f is the frictional energy lost per unit rotation of the flywheel and it is assumed to be steady. n_1 is the number of rotations completed by the flywheel, when the mass attached string has left the axle.

Even after the string has left the axle, the fly wheel continue to rotate and its angular velocity would decrease gradually and come to a rest when all is rotational kinetic energy has been used by the frictional energy. If n_2 is the number of rotation made by the flywheel after the string has left the axle,

$$n_2 f = \frac{1}{2} I\omega^2 \quad \text{or} \quad f = \frac{1}{2n_2} I\omega^2 \qquad (03\text{-}02)$$

By substituting eq. 03-02 for *f* in eq. 03-01, we get the expression for moment of inertia as,

$$I = \frac{2mgH - mv^2}{\omega^2 (1 + \frac{n_1}{n_2})} \qquad (03\text{-}03)$$

By taking $v = r\omega$ in eqn. 3.03 and rearranging the eq. 03-13, we can write down the expression for moment of inertial of flywheel as,

$$I = \frac{m[(2gH/\omega^2) - r^2]}{1 + (n_1/n_2)} \qquad (03\text{-}04)$$

Let t be the time taken by the flywheel to come to rest after the detachment of the mass. During this time interval, the angular velocity varies from ω to 0. So, the average angular velocity $\omega/2$ is,

$$\frac{\omega}{2} = \frac{2\pi n_2}{t} \quad \text{or} \quad \omega = \frac{4\pi n_2}{t} \qquad (03\text{-}05)$$

Procedure:

1. Setup the experiment as shown in Fig. 03-01 by taking a string of appropriate length and mass m.
2. Allow the string to unwind releasing the mass.
3. Count the number of rotation of the flywheel n_1 when the mass touches the ground.
4. Switch on the stopwatch when the moment the mass touches the ground and again count the number of rotation of flywheel, n_2 before it comes to rest. Stop the watch when the rotation ceases and note down the reading t.
5. Repeat the measurement for at least three times with the same string and mass such that n_1 , n_2 and t are closely comparable. Take their average value.
6. Repeat the measurement for another mass.
7. Measure the radius of axle using a vernier calipers and the length of the string using a scale.
8. Calculate the moment of inertia and maximum angular velocity ω using eq. 03-04 and 03-05

Observation Table

The suggested table is given as follows.

S.No.	m kg	n_1				n_2				t(s)			
		1st	2nd	3rd	Mean	1st	2nd	3rd	Mean	1st	2nd	3rd	Mean
1	Bare system												
2	Bare system + Regular body												

Calculations

(Students to fill in)

Results

The moment of inertia of given flywheel is ………………..

Precautions

(Students to fill in)

Sources of Error

(Students to fill in)

Reference

https://nptel.ac.in/courses/122103010/5)

LAB 04: To determine the Young's Modulus of a Wire by Optical Lever Method.

Aim

To determine Young's modulus of elasticity of the material of a given wire.

Objectives

To measure the Young's Modulus (Y) of a metal wire using an optical lever and a laser beam; to look at the relationship between the stress on a material and its resulting strain.

Apparatus

A meter stick, Laser, Optical Lever, Cu wire attached to micro-meter pivot, hanging masses.

Method

The linear stress/strain relationship is expected to hold only for cases of small strain (and consequently small stress). Thus we must be able to measure extremely small deflections with high precision (well beyond the capabilities of a simple meter stick. To accomplish this task we use an "optical lever" (a mirror mounted on a small pivot). In the figure below the mass M can be varied by stacking additional weights onto the hanger. As one adds these weights the Cu wire will stretch and the laser spot strike at a lower position on the far wall.

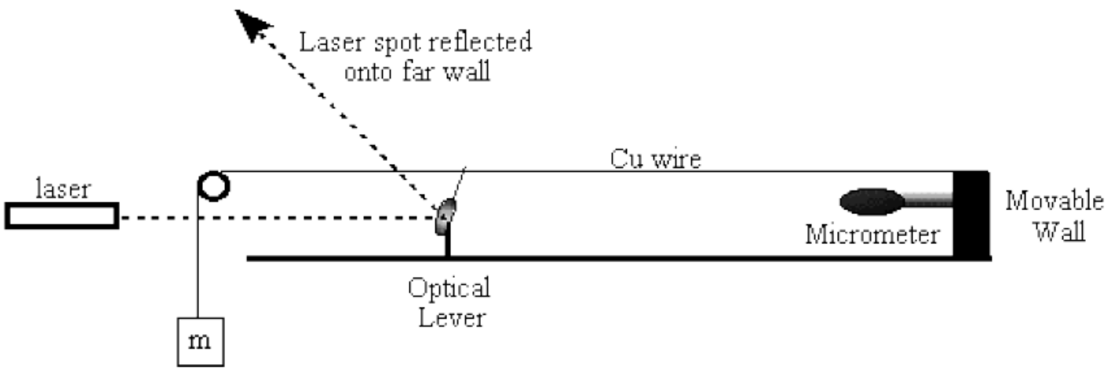

Fig 04-01: Experimental setup

The micrometer is used to move the wire a known distance while observing the laser spot. This allows for a calibration of the laser spot position in terms of the actual deflection of the wire.

PROCEDURE

1. Set up the system as shown in the diagram with the 50 gram "hanger" as your only weight.

CAUTION #1: NEVER stare directly into the laser beam.

CAUTION #2: NEVER place more than 300 g on the wire!

2. By turning the micrometer and/or moving the base of the optical lever, position the reflected beam spot on the far wall at a convenient height.

3. By turning the micrometer (don't change anything else), make the beam spot move vertically. Make a graph of micrometer reading (your TA will teach you to read the micrometer) vs. reflected beam spot position. Take at least 5 points and make the beam spot position vary by roughly one meter.

4. Find the slope, m, of the best straight line fit to your graph. From this point on, you can use the relation $D_l = m\, D_{spot}$, where D_l is the change in position of the top of the optical lever, and D_{spot} is the change in vertical position of the reflected spot.

Q1. Justify the relation given above: $D_l = m\, D_{spot}$.

5. Measure using the meter stick l_0, the distance from the "fixed" end of the copper wire to the point of contact with the optical lever.

Be careful not to "bump" the apparatus.

Q2. Why is the precision of this experiment not ruined by a measurement using a meter stick?

6. Using the diameter of the wire (provided by the TA), calculate the stress of the wire in proper MKS units.

7. Vary the stress in the wire by adding weights to the hanging mass. At each point record the total weight and the spot location.

Compute for each point the stress you applied and the strain you observed.

8. Make a plot of stress vs. strain.

Q3. Does your measurement support a linear stress strain relationship?

Q4. Calculate from your graph the Young's Modulus of copper.

Observations

(Students should fill in the details here)

Calculations

Precautions

Sources of error

References

http://superk.physics.sunysb.edu/~mcgrew/phy126/labs/126-01_Young_s_Modulus.pdf

http://cstl-csm.semo.edu/mlcobb/ph230/Master-PH230Cobb.pdf

LAB 05: TO DETERMINE THE MODULUS OF RIGIDITY OF A WIRE BY MAXWELL'S NEEDLE.

AIM

TO DETERMINE THE MODULUS OF RIGIDITY OF MATERIAL OF GIVEN WIRE BY DYNAMICAL METHOD USING MAXWELL NEEDLE..

Apparatus

Maxwell needle, stop watch, screw gauge, meter scale.

FORMULA:

The following formula is used for the determination of modulus of rigidity (η)

$$\eta = \frac{2\pi l}{r^4}\frac{(m_s - m_H)L^2}{\left(T_1^2 - T_2^2\right)}$$

Where *l*: length of wire,

L: length of Maxwell needle,

r: radius of wire,

m_S: mean mass of solid cylinders,

m_H; mean mass of hollow cylinders,

T_1: time period of oscillation when solid masses are outside,

T_2: time period of oscillation when solid cylinders are inside.

FIGURE OF EXPERIMENTAL SETUP

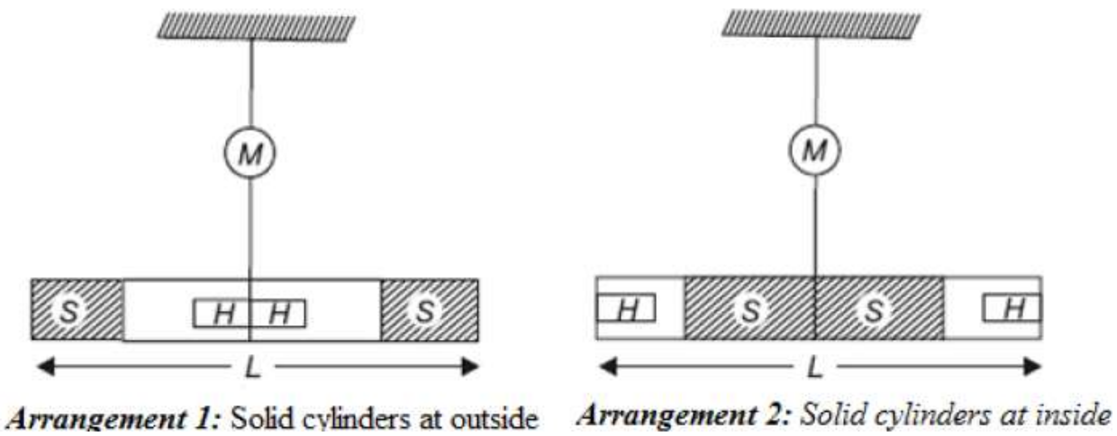

Fig 05-01: Experimental setup

PROCEDURE:

(1) Measure the length of wire using meter scale through which the Maxwell needle is hanged. This

will give you value of *l*.

(2) Measure the length of Maxwell needle using meter scale. This will give you value of L.

(3) Measure the mass of both solid cylinders using balance and do its half, this will provide the value of m_S.

(4) Measure the mass of the both hollow cylinders and do its half, this will provide the value of m_H.

(5) Find out the least count of screw gauge and zero error in it.

(6) Using screw gauge, measure the diameter of wire. Its half will provide the value of radius of wire.

(7) Find out the least count of stop watch.

(8) Now put the hollow cylinders at inside and solid cylinders at outside of the Maxwell needle. Oscillate it in horizontal plane about vertical axis. Note the time for 10, 20 and 30 oscillations. Divide the time with number of oscillations and find its mean. This will provide the value of T_1.

(9) Now place solid cylinders at inside and hollow cylinders at outside of the Maxwell needle. Oscillate it in horizontal plane about vertical axis. Note the time for 10, 20 and 30 oscillations. Divide the time with number of oscillations and find its mean. This will provide the value of T_2.

(10)Put all the value in given formula and solve it with log method.

OBSERVATIONS:

(1) Length of wire (*l*) =……..cm

(2) Length of Maxwell needle (L) =…….cm

(3) Mean mass of solid cylinders (m_S) =……..gm

(4) Mean mass of hollow cylinders (m_H) =……gm

(5) Least count of screw gauge =

Number of divisions on circular scale

pitch =……..cm

(6) Zero error in screw gauge=………..cm

(7) ***Table for diameter of wire***

Sr No	Reading in one direction (d1)			Reading in perpendicular direction (d2)			Measured
	Linear Scale Reading M (cm)	Circular Scale Reading (n)	Diameter d1 = M + n × Least Count (cm)	Linear Scale Reading M (cm)	Circular Scale Reading (n)	Diameter d2 = M + n × Least Count (cm)	Diameter (d1+d2)/2 (cm)

Mean diameter = ……………… cm

(8) *Radius of wire (r) = D/2 =……..cm*

(9) Least count of stop watch =……….. s

(10)

Sr No	No of Oscillations (n)	For outside solid cylinders			For inside solid cylinders		
		t_1 (s)	$T_1 = t_1/n$ (s)	Mean T_1 (s)	t_2 (s)	$T_2 = t_2/n$ (s)	Mean T_2 (s)

Calculations

Use the formula

$$\eta = \frac{2\pi\ell}{r^4}\frac{(m_s - m_H)L^2}{\left(T_1^2 - T_2^2\right)}$$

Put the values of the variables in the formula and find the value of η

Result

The value of modulus of rigidity is ………………………… N/m²

Precautions

1. There should not be any kinks in the wire.
2. The Maxwell needle should remain horizontal and should not vibrate up and down.
3. The amplitude of vibration/oscillation should be small so that wire is not twisted beyond the elastic limit.
4. To avoid the backlash error, the circular scale of screw gauge should be moved in one direction.

References

https://dkpandey.weebly.com/uploads/1/3/5/3/13534845/maxwell_needle.pdf

LABORATORY COMPONENT CREDIT 02

To determine the Elastic Constants of a Wire by Searle's method.

To determine g by Bar Pendulum.

To determine g by Kater's Pendulum.

To determine g and velocity for a freely falling body using Digital Timing Technique

To study the Motion of a Spring and calculate (a) Spring Constant (b) Value of g

LAB 06: TO DETERMINE THE ELASTIC CONSTANTS OF A WIRE BY SEARLE'S METHOD.

Aim

To determine Young's modulus of elasticity of the material of a given wire and thereby find other constants of elasticity.

Apparatus

Searle's apparatus, two long steel wires of same length and diameter, a metre scale, a screw gauge, eight 1/2 kg slotted weights and a 1 kg hanger.

Description of Searle's Apparatus

Construction: Searle's apparatus consists of two metal frames F_1 and F_2. Each frame has torsion head at the upper side and hook at the lower side. These frames are suspended from two wires AB and CD of same material, length and cross-section. The upper ends of the wires are screwed tightly in two torsion heads fixed in same rigid support.

A constant weight of 1 kilogram is suspended from the hook of the frame F_2 attached to the auxiliary wire CD, which keeps the wire taut. A hanger H of 1 kilogram weight is suspended from the hook of the other frame F_1. The experimental wire AB can be loaded by slipping slotted weights on the hanger.
A spirit level rests horizontally with its one end hinged in the frame F_2. The other end of the spirit level rests on the tip of a spherometer screw fitted in the frame F_1. The spherometer screw can be rotated up and down along a vertical pitch scale marked in millimetre. The two frames are kept together by cross bars E_1 and E_2.

Working: To perform the experiment, kinks are removed from the wire AB by loading and unloading it two or three times. All the weights are then removed from the hanger. The wire AB is kept taut under the weight of the hanger alone.
The spherometer screw is then rotated till the bubble comes in the middle of the spirit level. The spherometer disc reading is recorded for zero load.
A half kilogram weight is now slipped in the hanger. The wire AB extends and the frame F: moves down. The levelling is disturbed. The bubble is again brought in the middle by rotating the screw upwards. The distance by which screw is turned upwards gives the elongation of the wire due to half kilogram weight. A number of observations are taken by increasing the load on the hanger in steps of half kilogram each. The observations are then repeated by decreasing the load in the same order till all the weights are removed from the hanger. The mean of these observations is taken.
A graph is plotted between load M and mean extension l. It is a straight line. From the graph, mean increase in length l for a load M kg is found.

The measurements of length and radius of the wire are taken and value of Y is calculated from the relation,

$$Y = \frac{MgL}{\pi r^2 l} \text{ Nm}^{-2}.$$

Theory

If a wire of length L and radius r be loaded by a weight Mg and if l be the increase in length.

$$\text{Then, Normal stress} = \frac{Mg}{\pi r^2}$$

$$\text{and} \quad \text{Longitudinal strain} = \frac{l}{L}$$

$$\text{Hence, Young's modulus} = \frac{\text{Normal stress}}{\text{Longitudinal strain}}$$

$$\text{or} \qquad Y = \frac{Mg / \pi r^2}{l / L}$$

$$\text{or} \qquad Y = \frac{MgL}{\pi r^2 l}$$

Knowing L and r, and finding l for known Mg, Y can be calculated.

DIAGRAM

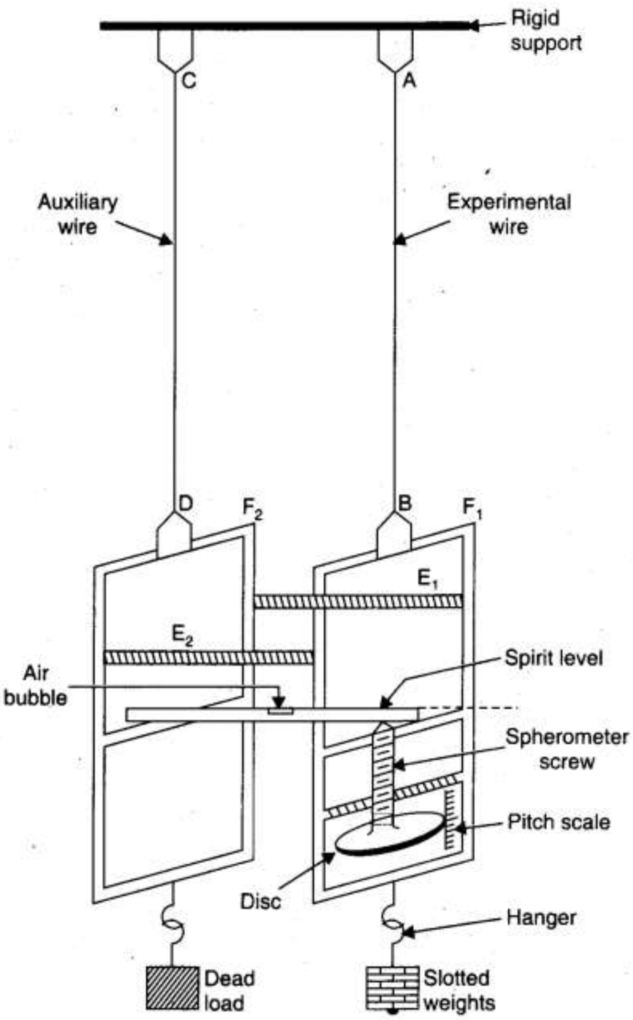

Fig. Searle's apparatus.

Procedure

1. Take two steel wires of same length and diameter and tight their ends in torsion screws A, B and C, D as shown in diagram. Wire AB becomes experimental wire and wire CD becomes auxiliary wire.
2. Suspend a 1 kg dead load from hook of frame F_2.
3. Suspend a 1 kg hanger and eight 1/2 kg slotted weights from hook of frame F_1. The experimental wire becomes taut.
4. Remove kinks from experimental wire by pressing the wire between nails of right hand thumb and first finger (through a handkerchief) and moving them along the length of the wire.

5. Remove all slotted weights from hanger. Now each wire is equally loaded with 1 kg weight.
6. Measure length of experimental wire from tip A to tip B using a metre scale.
7. Find the pitch and the least count of the screw gauge.
8. Measure diameter of the experimental wire at five different places, equally spaced along the length (near two ends, two quarter distance from ends and middle). At each place, measure diameter along two mutually perpendicular directions. Record the observations in the table.
9. Note the breaking stress for steel from table of constants. Multiply that by the cross section area of the wire to find breaking load of the wire. The maximum load is not to exceed one-third of the breaking load.
10. Find the pitch and the least count of the spherometer screw attached to frame F_1.
11. Adjust the spherometer screw such that the bubble in the spirit level is exactly in the centre. Note the reading of the spherometer disc. This reading is recorded against zero load.
12. Gently slip a 1/2 kg slotted weight in the hanger and wait for two minutes to allow the wire to extend fully. Bubble moves up from the centre.
13. Rotate the spherometer screw to bring the bubble back to centre. Note the reading of the spherometer disc. This reading is recorded against 1 kg load in load increasing column.
14. Repeat steps 12 and 13 till all the eight 1/2 kg slotted weights have been used (now total load on experimental wire is 5 kg which must be one-third of the breaking load).
15. Now remove one slotted weight (load decreasing), wait for two minutes to allow the wire to contract fully. Bubble moves down from the centre.
16. Repeat step 13. The reading is recorded against load in hanger in load decreasing column.
17. Repeat steps 15 and 16 till all the eight slotted weights are removed (now load on experimental wire is 1 kg the initial load).Observations for same load in load increasing and in load decreasing column must not differ much. Their mean is taken to eliminate the error.
18. Record your observations as given below.

Observations

Length of experimental wire AB, L = ………cm = ……….m.

Measurement of diameter of wire = ………..cm =………..m

Pitch of the screw gauge (p) = 0.1 cm

Number of divisions on the circular scale = 100

Least count of screw gauge (L.C.) =0.1/100 =0.001 cm.

Zero error of screw gauge (e) = …….cm.

Zero error of screw gauge (e) = -e =…….cm.

Table 1. Diameter of experimental wire

Serial No. of Obs.	Linear Scale Reading N (cm)	Circular Scale Reading: No. of division on reference line (n)	Circular Scale Reading: Value n × (L.C.) (cm)	Total Reading N + n × (L.C.) d (cm)
1. ⊖				d_1 =
1. ⊕				d_2 =
2. ⊖				d_3 =
2. ⊕				d_4 =
3. ⊖				d_5 =
3. ⊕				d_6 =
4. ⊖				d_7 =
4. ⊕				d_8 =
5. ⊖				d_9 =
5. ⊕				d_{10} =

Measurement for extension of wire

Breaking stress for steel (from table), B = ... N m^{-2}

Area of cross-section of wire, πr^2 = ... cm^2 = ... m^2

Breaking load, $= B\pi r^2$ = ... N

$$= \frac{B \cdot \pi r^2}{9.8} = \text{......} \text{ kg} \quad (\because \quad 1 \text{ kg} = 9.8 \text{ N})$$

1/3rd of breaking load = ... kg

Pitch of spherometer screw, (p) = 0.1 cm

Number of divisions in the disc = 100

Least Count of spherometer (L.C.) $= \frac{0.1}{100} = 0.001$ cm.

Table 2. load and extension

Serial No. of Obs.	Load on hanger M (kg)	Spherometer Screw Reading			Extension for load 2.5 kg
		Load increasing x (cm)	Load decreasing y (cm)	Mean $z = \frac{x+y}{2}$ (cm)	l (cm)
1.	0.0			$z_1 =$	
2.	0.5			$z_2 =$	
3.	1.0			$z_3 =$	
4.	1.5			$z_4 =$	
5.	2.0			$z_5 =$	
6.	2.5			$z_6 =$	$l_1 = (z_6 - z_1) =$
7.	3.0			$z_7 =$	$l_2 = (z_7 - z_2) =$
8.	3.5			$z_8 =$	$l_3 = (z_8 - z_3) =$
9.	4.0			$z_9 =$	$l_4 = (z_9 - z_4) =$

Calculations

From Table 1

Mean observed diameter of the wire,

$$d_0 = \frac{d_1 + d_2 + \dots\dots + d_{10}}{10} = \dots\dots \text{ cm.}$$

Mean corrected diameter of the wire,

$$d = (d_0 + c) = \dots\dots \text{ cm} = \dots\dots \text{ m.}$$

Mean radius of wire, $$r = \frac{d}{2} = \dots\dots \text{ m.}$$

From Table 2

Mean extension for 2.5 kg load

$$l = \frac{l_1 + l_2 + l_3 + l_4}{4} = \dots\dots \text{ cm} = \dots\dots \text{ m.}$$

From formula, $$Y = \frac{MgL}{\pi r^2 l} = \frac{2.5 \times 9.8 \times L}{\pi r^2 \times l} \text{ N m}^{-2}.$$

Calculation of M/l from graph

If we plot a graph between M and z taking M along X-axis and z along Y-axis, the graph comes to be a straight line. From it l for a known value of M can be calculated. The same value can be used to get a single average value of Young's modulus Y.

Result

1. The Young's modulus for steel as determined by Searle's apparatus $= \ldots\ldots \text{ N m}^{-2}$.

2. Straight line graph between load and extension shows that stress $\propto$ strain. This verifies Hooke's Law.

Percentage Error

Actual value of Y for steel (from tables) $= \ldots\ldots \text{ N m}^{-2}$

Difference in values $= \ldots\ldots \text{ N m}^{-2}$

$$\text{Percentage error} = \frac{\text{Difference in values}}{\text{Actual value}} = \ldots\ldots \%.$$

It is within limits of experimental error.

Calculate the other constants of elasticity using the conversion formula we learned in the discussion on elasticity.

Precautions

1. Both the wires (experimental and auxiliary) should be of same length, material and cross-section.
2. Both the wires should be supported from same rigid support.
3. Kinks should be removed from experimental wire before starting experiment.
4. Diameter of wire should be measured at different places and along two mutually perpendicular directions at every place.
5. Slotted weights should be added and removed gently.
6. Two minutes wait should be made after adding or removing a weight.
7. Load should be increased or decreased in regular steps.

Sources of error

1. Experimental wire may not have uniform cross-section throughout.
2. The slotted weights may not have standard weight.

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LAB 07: TO DETERMINE G BY BAR PENDULUM.

Aim

To determine the value of acceleration due to gravity and radius of gyration using bar pendulum.

Apparatus

Bar pendulum, stop watch and meter scale.

FORMULA:

A) The general formula of the time period for bar pendulum is given by following equation:

$$T = 2\pi\sqrt{\frac{\frac{k^2}{l}+l}{g}} = 2\pi\sqrt{\frac{l_2+l_1}{g}}$$

$$\boxed{g = \frac{4\pi^2(l_1+l_2)}{T^2} = \frac{4\pi^2 L}{T^2}}$$

(1)

Where l: distance between C.G. and suspension point, L: distance between suspension and oscillation points,

$$L = l_1 + l_2 = l + \frac{k^2}{l},$$

g: acceleration due to gravity, T: time period

B) The time period is minimum when $l = \pm k$, in this situation the equation (1) becomes as:

$$T_{min} = 2\pi\sqrt{\frac{2k}{g}}$$

or

$$\boxed{g = \frac{8\pi^2 k}{T_{min}^2}}$$

(2)

where, k: radius of gyration,

T_{min}: minimum time period.

The value of 'g' can be calculated using equations (1) and (2).

The values of L, T, k and T_{min} are obtained using graph between T and L for bar pendulum which is shown in following figure.

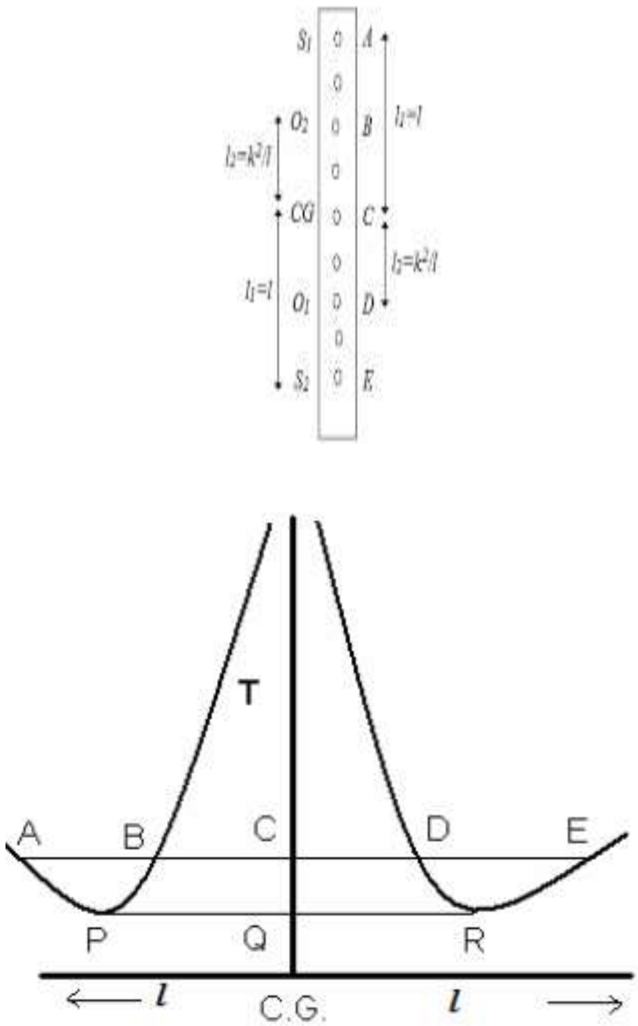

Fig 07-01: Reference figure for the experiment

(a) $L_1=AC+CD$, $L_2=EC+CB$ *and* $L=(L_1+L_2)/2$, T=*time at C*

(b) k =(PQ+QR)/2 and T_{min}= time at Q

C) The radius of gyration can be obtained with following formula

$$\boxed{k = \sqrt{l_1 l_2}} \qquad (3)$$

Where $l_1=(AC+CE)/2$, $l_2=(BC+CD)/2$

THEORY

A simple pendulum consists of a small body called a “bob” (usually a sphere) attached to the end of a string the length of which is great compared with the dimensions of the bob and the mass of which is negligible in comparison with that of the bob. Under these conditions the mass of the bob may be regarded as concentrated at its center of gravity, and the length l of the pendulum is the distance of this point from the axis of suspension. When a simple pendulum swings through a small arc, it executes linear simple harmonic motion of period T, given by the equation

$$T = 2\pi \sqrt{(l/g)} \qquad (1)$$

where g is the acceleration due to gravity. This relation-ship affords one of the simplest and most satisfactory methods of determining g experimentally.

When the dimensions of the suspended body are not negligible in comparison with the distance from the axis of suspension to the center of gravity, the pendulum is called a compound, or physical pendulum. Any object mounted upon a horizontal axis so as to vibrate under the force of gravity is a compound pendulum. The motion of such a body is an angular vibration about the axis of suspension. The expression for the period of a compound pendulum may be deduced from the general expression for the period of any angular simple harmonic motion

$$T = 2\pi \sqrt{\frac{\theta}{a}} \qquad (2)$$

and the application of Newton’s second law of motion for rotating bodies

$$\tau = I\,\alpha$$

where θ is the angular displacement,

α the angular acceleration,

τ the torque and

I the rotational inertia of the body.

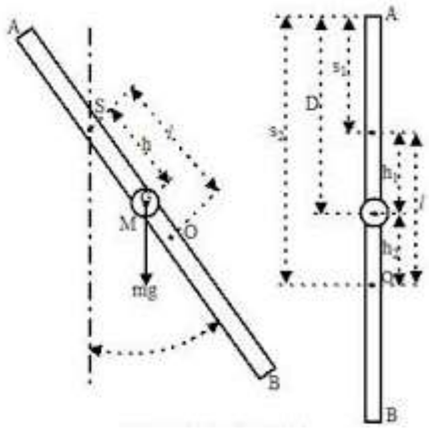

Fig 07-02 Compound pendulum

Figure 07-02 represents a compound pendulum of mass m, consisting of a rectangular bar AB to which a cylindrical mass M is attached. The pendulum is suspended on a transverse axis through the point S. In the diagram, the cylindrical mass M is represented as being exactly in the middle of the bar, thereby making a symmetrical system in which the center of gravity G is at the geometrical center. Obviously, this particular condition is a very special case, and has nothing whatever to do with the general treatment of the problem.

In the equilibrium position, the center of gravity G is vertically below the axis of suspension S. When the body is rotated through an angle θ, the weight of the system mg, which may be regarded as concentrated at the center of gravity, exerts a restoring torque about S given by

$\tau = mgh \sin\theta$ (4)

where h is the distance from the axis of suspension to the center of gravity. If a minus sign is used to indicate the fact, that torque τ is opposite in sign to the displacement θ, Eqs. (3) and (4) yield

$Ia = -mgh \sin\theta$ (5)

When the angular displacement θ is sufficiently small, sinθ is approximately equal to θ measured in radians. With this restriction Eq. (5) may be written

$a = ((-mgh)/I)\theta$ (6)

Since m, g, h and I are all numerically constant for any given case, Eq. (6), may be written simply

$a = -k\theta$ (7)

where k is a constant. Equation (7) is the defining equation of angular simple harmonic motion, i.e., motion in which the angular acceleration is directly proportional to the angular displacement and oppositely directed. Since the system executes angular simple harmonic motion, substitution of the expression for a from Eq. (6) in Eq. (2) yields the equation for the period of a compound pendulum

$T = 2\pi \sqrt{(I/mgh)}$ (8)

where I is the rotational inertia of the pendulum about the axis of suspension S. It is convenient to express I in terms of I_0, the rotational inertia of the body about an axis through its center of gravity G. If the mass of the body is m,

$I_o = mk_o^2$ (9)

where k_o is the radius of gyration about an axis through G. For any regular body, k_o can be computed by means of the appropriate formula (see any handbook of physics or engineering); for an irregular body it must be determined experimentally. The rotational inertia about any axis parallel to the one through the center of gravity is given by

$I = I_o + mh^2$ (10)

where h is the distance between the two axes. Substitution of the relationships of Eqs. (9) and (10) in Eq. (8) yields

$T = 2\pi \sqrt{((k_o^2 + h^2)/gh)}$ (11)

This equation expresses the period in terms of the geometry of the body. It shows that the period is independent of the mass, depending only upon the distribution of the mass (as measured by k_o) and upon the location of the axis of suspension (as specified by h). Since the radius of gyration of any given body is a constant, the period of any given pendulum is a function of h only. Comparison of Eq. (1) and Eq. (11) shows that the period of a compound pendulum suspended on an axis at a distance h from its center of gravity is equal to the period of a simple pendulum having a length given by

$$l = (k_o^2 + h^2)/h = h + (k_o^2/h) \quad (12)$$

The simple pendulum whose period is the same as that of a given compound pendulum is called the "equivalent simple pendulum."

It is sometimes convenient to specify the location of the axis of suspension S by its distance s from one end of the bar, instead of by its distance h from the center of gravity G. If the distances s_1, s_2 and D (Fig. 1) are measured from the end A, the distance h_1 must be considered negative, since h is measured from G. Thus, if D is the fixed distance from A to G, $s_1 = D + h_1$, $s_2 = D + h_2$ and, in general, $s = D + h$. Substitution of this relationship in Eq. (11) yields

$$T = 2\pi \sqrt{((k_o^2 + (s - D)^2)/(g(s - D)))} \quad (13)$$

The relationships between T and s expressed by Eq. (l3) can best be shown graphically. When T is plotted as a function of s, a pair of identical curves SPQ and S'P'Q' are obtained as illustrated in Fig. 2. (The dotted portions represent extrapolations over apart of the body where it is difficult to obtain experimental data with this particular pendulum.) Analysis of these curves reveals several interesting and remarkable properties of the compound pendulum. Beginning at the end A, as the axis is displaced from A toward B the period diminishes, reaching a minimum value at P, after which it increases as S approaches G. The two curves are asymptotic to a perpendicular line through G, indicating that the period becomes infinitely great for an axis through the center of gravity. As the axis is displaced still farther from A (to the other side of G), the period again diminishes to the same minimum value at a second point P', after which it again increases.

A horizontal line SS', corresponding to a chosen value of T, intersects the, graph in four points, indicating that there are four positions of the axis, two on each side of the center of gravity, for which the periods are the same. These positions are symmetrically located with respect to G. There are, therefore, two numerical values of h for which the period is the same, represented by h_1 and h_2 (Figs. 1 and 2). Thus, for any chosen axis of suspension S there is a conjugate point O on the opposite side of the center of gravity such that the periods about parallel axes through S and O are equal. The point O is called the center of oscillation with respect to the particular axis of suspension S.

Consequently, if the center of oscillation of any compound pendulum is located, the pendulum may be inverted and supported at O without altering its period. This so-called reversibility is one of the unique properties of the compound pendulum and one that has been made the basis of a very precise method of measuring g (Kater's reversible pendulum).

It can be shown that the distance between S and O is equal to l, the length of the equivalent simple pendulum. Equating the expressions for the squares of the periods about S and O, respectively, from Eq. (11)

$T^2 = (4\pi^2/g)((k_o^2 + h_1^2)/h_1) = (4\pi^2/g)((k_o^2 + h_2^2)/h_2)$ (14)

from which

$T^2 = (4\pi^2/g)\ (h_1 + h_2)$ (15)

or $T = 2\pi\sqrt{((h_1 + h_2)/g)}$ (16)

Comparison of Eqs. (1) and (16) shows that the length l of the equivalent simple pendulum is
$l = h_1 + h_2$ (17)

Thus, the length of the equivalent simple pendulum is SO (Figs. 1 and 2).

S' and O' are a second pair of conjugate points symmetrically located with respect to S and O respectively, i.e., having the same numerical values of h_1 and h_2.

Further consideration of Fig. 2 reveals the fact that the period of vibration of a given body cannot be less than a certain minimum value T_o, for which the four points of equal period reduce to two, S and O' merging into P, and S' and O merging into P' as h_1 becomes numerically equal to h_2. The value of h_o corresponding to minimum period can be deduced by solving Eq. (14) for k_o^2, which yields

$k_o^2 = h_1h_2$ (18)
and setting
$h_o = h_1 = h_2$ (19)

Thus

$h_o = k_o$ (20)

Substituting in Eq. (12) yields

$l_o = 2k_o$ (21)

Thus, the shortest simple pendulum to which the compound pendulum can be made equivalent has a length l_o equal to twice the radius of gyration of the body about a parallel axis through the center of gravity. This is indicated in Fig. 2 by the line PP'. Inspection of the figure further shows that, of the two values of h for other than minimum period, one is less than and one greater than k_o. From the foregoing it is evident that if two asymmetrical points S and O can be found such that the periods of vibration are identical, the length of the equivalent simple pendulum is the distance between the two points, and the necessity for locating the center of gravity is eliminated. Thus, by making use of the reversible property of the compound pendulum, simplicity is, achieved similar to that of the simple pendulum, the experimental determinations being reduced to one measurement of length and one of period.

PROCEDURE:

(1) Place the knife-edges at the first hole of the bar.
(2) Suspend the pendulum through rigid support with the knife-edge.
(3) Oscillate the pendulum for small amplitude (θ=3~4).
(4) Note the time taken for 20 oscillations and measure the distance of the hole from the C.G.of the bar.
(5) Repeat the observations (2)-(4) for knife-edges at first half side holes of bar.
(6) Repeat the process (1)-(5) for the second half side of the bar.
(7) Plot the graph between T and L.

OBSERVATIONS:

1. Least count of the stop watch = ….. sec
2. Least count of the meter scale =…… cm

Sr No	Length l (cm)	Total Time (t) taken for 20 osc. (s)	Time period T = t/20 (s)
First Half side of bar			
1	45		
2	40		
3	35		

4	30		
5	25		
6	20		
7	15		
8	10		
9	5		
Second Half side of bar			
1	-5		
2	-10		
3	-15		
4	-20		
5	-25		
6	-30		
7	-35		
8	-40		
9	-45		

Plot the graph of T vs L. (attach)

CALCULATIONS

From the graph of T vs L,

L = (AD+EB)/2=…………………,

T = …………………… s,

k = PR/2 = ……………..

Tmin = ………………… s.

$$l_1 = \frac{AC + CE}{2} = \cdots \ldots \ldots \ldots \ldots .. \; cm$$

$$l_2 = \frac{BC + CD}{2} = \cdots \ldots \ldots \ldots \ldots .. \; cm$$

$$k = \sqrt{l_1 \times l_2} = \cdots \ldots \ldots \ldots \ldots \ldots \ldots \ldots cm$$

$$g_1 = \frac{4\,\pi^2 \mathrm{L}}{T^2} = \cdots \ldots \ldots \ldots \ldots .. \; cm\, s^{-2}$$

$$g_2 = \frac{8\,\pi^2 \mathrm{k}}{T_{min}^2} = \cdots \ldots \ldots \ldots \ldots .. \; cm\, s^{-2}$$

$$g = \frac{g_1 + g_2}{2} = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots..cm\, s^{-2}$$

RESULTS

1. Acceleration due to gravity = …………………cm s^{-2}

2. Radius of gyration (k) = ………………… cm (by calculation from l_1 and l_2)

= …………………… cm (from plot)

PRECAUTIONS

1. The motion of the pendulum should be in a vertical plane. While taking the time, start taking observations after two oscillations to avoid any irregularity of motion.
2. The amplitude of oscillation should be small.

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LAB 08 TO DETERMINE G BY KATER'S PENDULUM.

AIM

To determine g, the acceleration due to gravity at a particular location..

APPARATUS

Kater's pendulum, stopwatch, meter scale and knife edges.

THEORY

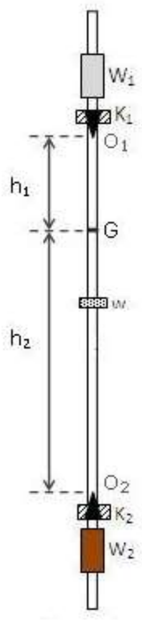

Fig 08-01: Kater's pendulum

Kater's pendulum, shown in Fig. 08-01, is a physical pendulum composed of a metal rod 1.20 m in length, upon which are mounted a sliding metal weight W_1, a sliding wooden weight W_2, a small sliding metal cylinder w, and two sliding knife edges K_1 and K_2 that face each other. Each of the sliding objects can be clamped in place on the rod. The pendulum can be suspended and set swinging by resting either of the knife edge on a flat, level surface. The wooden weight W_2 is the same size and shape as the metal weight W_1. Its function is to provide as near as possible, equal air resistance to swinging in either suspension, which happens if W_1 and W_2, and separately K_1 and K_2, are constrained to be equidistant from the ends of the metal rod. The centre of mass G can be located by balancing the pendulum on an external knife edge. Due to the difference in mass between the metal and wooden weights W_1 and W_2, G is not at the centre of the rod, and the distances h_1 and h_2 from G to the suspension points O_1 and O_2 at the knife edges K_1 and K_2 are not equal. Fine adjustments in the position of G, and thus in h_1 and h_2, can be made by moving the small metal cylinder w.

In Fig. 1, we consider the force of gravity to be acting at G. If h_i is the distance to G from the suspension point O_i at the knife edge K_i, the equation of motion of the pendulum is

$$I_i\ddot{\theta} = -Mgh_i \sin\theta$$

where I_i is the moment of inertia of the pendulum about the suspension point O_i, and i can be 1 or 2. Comparing to the equation of motion for a simple pendulum

$$Ml_i^2\ddot{\theta} = -Mgl_i \sin\theta$$

we see that the two equations of motion are the same if we take

$$Mgh_i / I_i = g / l_i \qquad (1)$$

It is convenient to define the radius of gyration of a compound pendulum such that if all its mass M were at a distance from O_i, the moment of inertia about O_i would be I_i , which we do by writing

$$I_i = Mk_i^2$$

Inserting this definition into equation (1) shows that

$$k_i^2 = h_i l_i \qquad (2)$$

If I_G is the moment of inertia of the pendulum about its centre of mass G, we can also define the radius of gyration about the centre of mass by
writing

$$I_G = Mk_G^2$$

The parallel axis theorem gives us

$$k_i^2 = h_i^2 + k_G^2 ,$$

so that, using (2), we have

$$l_i = \frac{h_i^2 + k_G^2}{h_i}$$

The period of the pendulum from either suspension point is then

$$T_i = 2\pi\sqrt{\frac{l_i}{g}} = 2\pi\sqrt{\frac{h_i^2 + k_G^2}{gh_i}} \qquad (3)$$

Squaring (3), one can show that

$$h_1T_1^2 - h_2T_2^2 = \frac{4\pi^2}{g}\left(h_1^2 - h_2^2\right)$$

and in turn,

$$\frac{4\pi^2}{g} = \frac{h_1T_1^2 - h_2T_2^2}{(h_1+h_2)(h_1-h_2)} = \frac{T_1^2 + T_2^2}{2(h_1+h_2)} + \frac{T_1^2 - T_2^2}{2(h_1-h_2)}$$

which allows us to calculate g,

$$g = 8\pi^2\left[\frac{T_1^2 + T_2^2}{h_1+h_2} + \frac{T_1^2 - T_2^2}{h_1-h_2}\right]^{-1} \qquad (4)$$

APPLICATIONS

Pendulums are used to regulate pendulum clocks, and are used in scientific instruments such as accelerometers and seismometers. Historically they were used as gravimeters to measure the acceleration of gravity in geophysical surveys, and even as a standard of length. The problem with using pendulums proved to be in measuring their length.

A fine wire suspending a metal sphere approximates a simple pendulum, but the wire changes length, due to the varying tension needed to support the swinging pendulum. In addition, small amounts of angular momentum tend to creep in, and the centre of mass of the sphere can be hard to locate unless the sphere has highly uniform density. With a compound pendulum, there is a point called the centre of oscillation, a distance l from the suspension point along a line through the centre of mass, where l is the length of a simple pendulum with the same period. When suspended from the centre of oscillation, the compound pendulum will have the same period as when suspended from the original suspension point. The centre of oscillation can be located by suspending from various points and measuring the periods, but it is difficult to get an exact match in the period, so again there is uncertainty in the value of l. With equation (4), derived by Friedrich Bessel in 1826, the situation is improved. $h_1 + h_2$, being the distance between the knife edges, can be measured accurately. $h_1 - h_2$ is more difficult to measure accurately, because accurate location of the centre of mass G is difficult. However, if T_1 and T_2 are very nearly equal, the second term in (4) is quite small compared to the first, and $h_1 - h_2$ does not have to be known as accurately as $h_1 + h_2$.

Kater's pendulum was used as a gravimeter to measure the local acceleration of gravity with greater accuracy than an ordinary pendulum, because it avoids having to measure l. It was popular from its invention in 1817 until the 1950's, when it began to be possible to directly measure the acceleration of gravity during free fall using a Michelson interferometer. Such an absolute gravimeter is not particularly portable, but it can be used to accurately calibrate a relative gravimeter consisting of a mass hanging from a spring adjacent to an accurate length scale. The relative gravimeter can then be carried to any location where it is desired to measure the acceleration of gravity.

PROCEDURE

- Shift the weight W_1 to one end of katers pendulum and fix it. Fix the knife edge K_1 just below it.
- Keep the knife edge K_2 at the other end and fix the wooden weight W_2 symmetrical to other end. Keep the small weight 'w' near to centre.
- Suspend the pendulum about the knife edge 1 and take the time for about 10 oscillations. Note down the time t_1 using a stopwatch and calculate its time period using equation $T_1 = t_1/10$.
- Now suspend about knife edge K_2 by inverting the pendulum and note the time t_2 for 10 oscillations. Calculate T_2 also.
- If $T_2 \neq T_1$, adjust the position of knife edge K_2 so that $T_2 \approx T_1$.
- Balance the pendulum on a sharp wedge and mark the position of its centre of gravity. Measure the distance of the knife-edge K_1 as h_1 and that of K_2 as h_2 from the centre of gravity.

OBSERVATIONS AND CALCULATIONS

To determine T_1 and T_2

Knife Edge	Time for 10 osc			Time Period
	Reading 1 (s)	Reading 2 (s)	Mean (s)	t/10 (s)
K1				
K2				

Distance of K_1 from C.G,h_1 = ...m.

Distance of K_2 from C.G,h_2 =...............m.

$$g = \frac{8\pi^2}{\frac{T_1^2 + T_2^2}{h_1 + h_2} + \frac{T_1^2 - T_2^2}{h_1 - h_2}}$$

Acceleration due to gravity, g =...............ms^{-2}.

RESULT

The acceleration due to gravity at a given place is found to be =..............ms^{-2}.

REFERENCES

http://vlab.amrita.edu/?sub=1&brch=280&sim=518&cnt=1

https://people.emich.edu/jthomsen/p332/expt1.pdf

LAB 09 TO DETERMINE G AND VELOCITY FOR A FREELY FALLING BODY USING DIGITAL TIMING TECHNIQUE

AIM

To determine acceleration due to gravity and velocity for free fall using digital timing.

APPARATUS

- Release mechanism (may be electromagnetic)
- Trip switch (hinged flap)
- Power supply, low voltage, DC
- Switch, SPDT
- Ball bearing ball, steel/iron (magnetic)
- Retort stand and boss
- Electronic timer
- Leads, 4 mm

APPARATUS

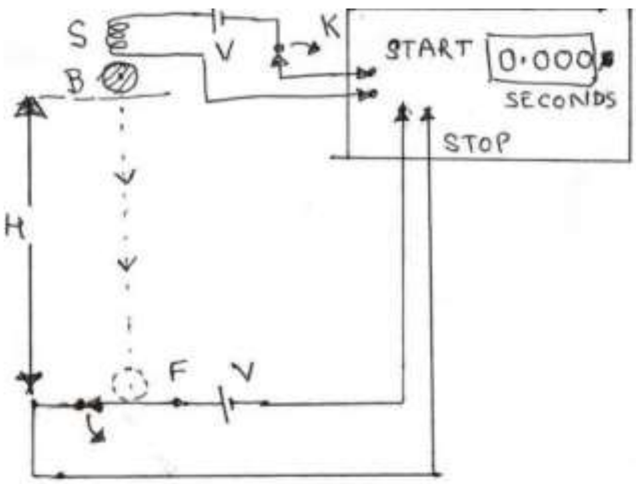

Fig 09-01: Experimental arrangement

The experimental arrangement consists of a solenoid S, a ball B of magnetic material (steel or iron), a switch K (which is normally "on" and when you press it, it will break the electricity), a timer which has two signal inputs called "start" and "stop" signal inputs, a flap F which get open when ball strikes it and causes electricity to break, and a power supply. The height from flap to the bottom of the ball (before release) is H as shown in Fig 09-01.

Before the beginning of the experiment, the power supply to the electromagnet is turned on and flap is covered. The timer is stopped and shows reading 000 ms.

When the switch is operated, it causes the current in electromagnet to be discontinued, thus the magnetic force which was holding the ball is withdrawn and hence the ball falls under the influence of gravity. The timer receives a signal from the circuit to ‘start’ the timer. The timer starts counting milliseconds.

The ball falls on the flap, which gets open due to the fall, it breaks the circuit and passes the signal to the timer which says ‘stop’. The timer stops. The time elapsed between Star and Stop is seen on the timer.

You can measure the height from which the ball got dropped. You should take the bottom of the ball as the reference point (not the centre).

PROCEDURE

1. Set up the apparatus as shown in the diagram. You may need to adjust the distance of fall and the point at which the ball strikes the flap.
2. Arrange the timer so that it *starts* when the electromagnet is switched off

   and *stops* when the hinged flap opens.
3. Check that the flap does open when the ball strikes it. You may need to make the distance of fall larger, or move the flap so that the ball strikes it further from the hinge.
4. Measure the distance *h* from the bottom of the ball to the hinged flap. Be careful to avoid parallax error in this measurement.
5. Measure the fall time three times and find the average.

   Repeat step **5** for a range of heights between 0.5 m and 2.0 m.

   Plot a graph of 2*h* against t^2.
6. Use the graph to find *g*.

OBSERVATIONS

Sr No	Height h (cm)	Time T (s)	X t^2 (s^2)	Y 2 h (cm)
1				
2				
3				
4				
5				
6				

7				
8				
9				
10				

Plot the relation between 2h versus t^2.(attach)

Since $h = \frac{1}{2}gt^2$, hence $2h = gt^2$ the slope of the graph will give you the value of g.

RESULT

The value of g as observed from the graph is ………. cm s^{-2}

PRECAUTIONS

1. Take due care not to get shock from electric equipments.
2. Take care that the flap should not be broken by the shock from falling ball.
3. Take due care that the ball does not fall on the body of the student.

REFERENCES

http://practicalphysics.org/measurement-g-using-electronic-timer.html

LAB 10: To study the Motion of a Spring and calculate (a) Spring Constant (b) Value of g

Aim

To determine the spring constant (restoring force per unit extension) of a spiral spring by methods of statics and dynamics and also to determine the acceleration due to gravity.

Apparatus

A spiral spring, 10 gm. weights- 5 No., a scale pan, a pointer and a stop watch.

Formula

Method of dynamics gives

$$K = \frac{4\pi^2(M_1 - M_2)}{(T_1^2 - T_2^2)}$$

Here K = Spring constant (restoring force per unit extension), T_1 is the time period of oscillation when mass of M_1 is in pan, T_2 is the time period with mass M_2 on the pan.

Method of statics gives

$$K = \frac{Mg}{l}$$

From this one can determine g as

$$g = K \times \frac{l}{M}$$

Apparatus

Let us know about the apparatus. The adjacent figure shows a mass spring system. A spiral spring whose restoring force per unit extension is to be determined is suspended from a rigid support as shown in the figure 10-01. At the lower end of the spring, a small scale-pan is fastened. A small horizontal pointer is also attached to the scale pan. A scale is also set in front of the spring in such a way that when spring vibrates up and down, the pointer freely moves over the scale.

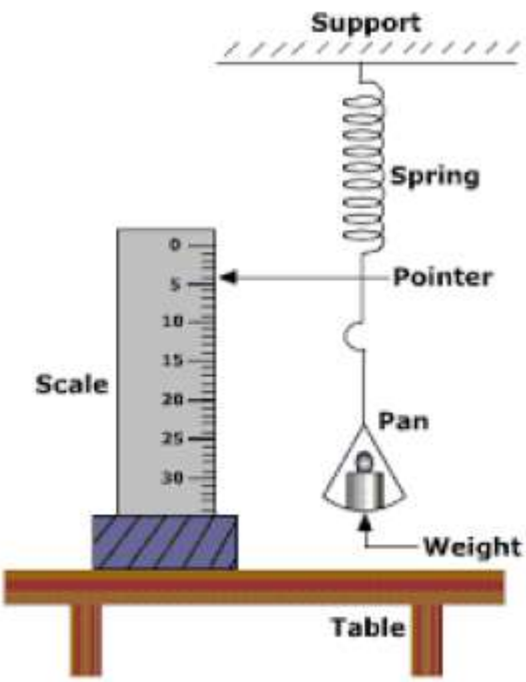

Fig 10.01: Apparatus for finding spring constant

Procedure

Let us perform the experiment to determine the restoring force per unit extension of a spiral spring and mass of the spring. We shall perform the two methods in the following way-

(A) Method of Dynamics

(i) Put gently a load M_1 (say 10gm.) in the pan. Now let us displace the pan vertically downward through a small distance and release it. You will see that the spring starts to perform simple harmonic oscillations.

(ii) With the help of stop watch, note down the time of a number of oscillations (say 10). Now get the time period or the time for one oscillation T_1 by dividing the total time by the total number of oscillations.

(iii) Increase the load in the pan to M2 (say 20 gm.). As described above, find out the time period T_2 for this load.

(iv)You may repeat the experiment with different values of load.

Record the observations in the table

(B) Method of Statics

(Note: Some people call it Statical Method erroneously. The method of statics is used for situations where randomness plays important role and rely on theory of probability and random variables. In the current situation, there is hardly any chance or random variables. The method of statics refers to the branch of mechanics called statics (other branch being dynamics) where forces are balanced in equilibrium and hence there are no motions.)

(i) Without no load in the scale-pan, note down the zero reading of the pointer on the scale.

(ii) Now place gently 10 gm. load (weight) in the pan. Stretch the spring slightly and the pointer moves down on the scale. In this steady position, note down the reading of the pointer. The difference of the two readings is the extension of the spring for the load in the pan.

(iii) Let us increase the load in the pan in equal steps until maximum permissible load is reached and note down the corresponding pointer readings on the scale.

(iv)The experiment is repeated with decreasing weights (loads).

OBSERVATIONS

(A) METHOD OF DYNAMICS: THE MEASUREMENT OF PERIODS T_1 AND T_2 FOR LOADS M_1 AND M_2

Least count of stop-watch = ...sec.

S.No.	Load in pan (gm.)		No. of oscillations	Time taken with load		Time period (sec)	
	M_1	M_2		M_1 sec	M_2 sec	T_1	T_2
1	10	20					
2	30	40					
3	50	60					
4	70	80					
5	90	100					

(B) Method of Statics: The measurement of extension of the spring

S.No.	Load(weight) in the pan (gm.)	Reading of pointer on the scale (meter)			Extension for 30 gm. (meter)	Mean extension (meter)
		Load increasing	Load decreasing	Mean		
1	10				(3)-(1)=	
2	20					
3	30				(4)-(2)=......	
4	40					
5	50				(5)-(3)=.......	

Calculations

Restoring force per unit extension of the spring-

$$K = \frac{4\pi^2(M_1 - M_2)}{(T_1^2 - T_2^2)}$$

= .... Newton/meter

Similarly, you should calculate K for other sets and then obtain the mean value.

After you have arrived at the value of K, you can find the value of acceleration due to gravity using the relation:

$$g = K \times \frac{l}{M}$$

$$g = \cdots \ldots \ldots \ldots \ldots cm/s^2$$

Results

(1) Spring Constant = ……………………….. (student should write value found with appropriate units)

(2) Acceleration due to gravity =……………………… (student should write value found with appropriate units)

Precautions and Sources of Errors:

(I) Dynamics Method

(1) The spring should oscillate vertically.

(2) The amplitude of oscillations should be small.

(3) Time periods T_1 and T_2 should be measured very accurately.

(II) Statics Method:

(1) The axis of the spring must be vertical.

(2) The spring should not be stretched beyond elastic limits.

(3) The pointer should move freely on the scale.

(4) Load (weight) should be placed gently in the scale pan.

(5) The scale should be set vertical. It should be arranged in such a way that it should give almost the maximum extension allowable.

(6) Readings should be taken very carefully from the front side.

References

http://uou.ac.in/sites/default/files/slm/BSCPH-104.pdf